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SAGA, A COMPUTER PROGRAM FOR SHORT ARC GEODETIC ADJUSTMENT OF SATELLITE OBSERVATIONS

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ABSTRACT

The program SAGA exploits short arc orbital constraints in effecting the adjustment of observations made by geodetic tracking nets embracing both optical systems (e.g., PC-1000, MOTS) and electronic ranging systems (Lasers, Secor, Geoceiver, etc.). Provisions are made for consideration of:

a) random errors in the observations and in the timing of observations;
b) serially correlated errors in observations;
c) errors in the adopted location of the center of mass;
d) systematic errors governed by error models having coefficients subject to a priori constraints.

The overall tracking net can include an indefinitely large number of stations (many hundreds) as long as no more than fifteen participate successfully in the observations of any pass. All orbital state vectors are treated as unknown and no limits are set on the number of state vectors that can be solved for simultaneously. Allowances are made in optical error modelling for reinitialization of error coefficients that becomes necessary when any station exposes more than one plate on a given pass. In the case of electronic tracking, up to three dropouts in tracking can be accommodated for each station on each pass with appropriate reinitialization of error coefficients. A maximum of over 250 error coefficients can be exercised in the reduction of each pass. This becomes computationally feasible by virtue of algorithms providing the solution to problems in what is termed second order partitioned regression.

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1.0 INTRODUCTION

In previous series of investigations conducted by DBA Systems for AFCRL (Brown, Bush, Sibol (1963), (1964); Brown (1964b), we developed most of the theoretical framework for the present undertaking, namely the development of a general and advanced computer program for short arc geodesy. This program, hereafter referred to as SAGA (Short Arc Geodetic Adjustment), has drawn on and benefitted from experience gained from three preceding programs that were also based on the theoretical development referred to above. These predecessors consist of the following programs developed by DBA:

(1) GDAP (GEOS Data Adjustment Program) developed for NASA Goddard during the period 1966-1967 for the short arc reduction of observations of geodetic satellites (Lynn, 1967);

(2) MCT (Method of Continuous Traces) developed for AFCRL during the period 1966-1967 for the recovery of geodetic positions from measurements of sun-illuminated passive satellites recorded against the stellar background (Brown, Trotter, 1967);

(3) NAP (Tracking Network Analysis Program) developed for NASA Goddard during the period 1967-1969 for long arc orbital reduction, terrestrial, lunar, or interplanetary, with options for recovery of station coordinates and error model coefficients (Lynn, et al, 1969).

Properties of SAGA relative to GDAP, MCT and NAP are indicated in Table 1. All four programs are capable of exercising short arc orbital constraints in an unlimited multi-epoch mode (that is, the number of state vectors that can be solved for simultaneously is without present limit). In addition, NAP can exercise long arc constraints and can accommodate extraterrestrial orbits (it employs a general n body integrator capable of integrating through

	Feature	CDAP	MCT	NAP	SAGA
1.	Short arc constraints (unlimited no. epochs)	X	X	X	X
2.	Long arc constraints (unlimited no. epochs)			X	
3.	Extraterrestrial orbits			X	
4.	Geometric option (orbital constraints not used)	X			
5.	Recovery of station locations				
	(a) limited number	X		X	
	(b) unlimited number		X		X
6.	Recovery of coefficients of potential function			X(1)	
7.	Recovery of coordinates of center of mass			X(1)	X
8.	Recovery of tracking error coefficients				
	(a) optionally reinitialized after every pass	X	X	X	X
	(b) optionally stable over specified sets of passes	X		X	
9.	Consideration of random timing error				X
10.	Option for consideration of serially correlated errors by Autoregressive Feedback			X	X
11.	Solution of general normal equations by First Order Partitioned Regression	X	X		
12.	Solution of general normal equations by Second Order Partitioned Regression			X	X
13.	Tracking systems accommodated				
	(a) Optical (PC-1000, MOTS, Baker-Nunn, BC-4, etc.)	X	X	X	X
	(b) Electronic Ranging (Lasers, SECOR, GRARR, Radars)	X		X	X
	(c) Microwave Interferometer (MINITRACK)	X		X	
	(d) Noncumulative, one way doppler (TRANET)	X		X	
	(e) Cumulative, one way doppler (GEOCEIVER)				X
14.	Structure of program and organization of input/output optimized primarily for				
	(a) Recovery of station locations		X		X
	(b) Recovery of precise orbits			X	
	(c) Recovery of tracking error coefficients	X			
15.	Program operational on				
	(a) CDC 3100/3200 (16K core, 4 mag. tape units)	X	X		
	(b) CDC 3800 (Naval Research Lab.)	X			
	(c) UNIVAC 1230 (NASA Goddard)	X			
	(d) IBM 360/75/91/95 (NASA Goddard)	X		X	
	(e) IBM 7044/7094 (AFCRL)		X		X

(1) Extension underway to add indicated capability.

spheres of influence while allowing the gravitational potential of the currently dominant center of attraction to be represented by solid spherical harmonics up to degree and order $(n,m) = (24,24)$. While NAP can be employed for satellite geode··· its primary intent and organization are directed toward precise determination of orbits with appropriate consideration being given to (a) serially correlated errors of tracking systems, (b) systematic errors of tracking systems, (c) errors in locations of tracking stations and (d) errors in coefficients of the potential function (at this writing the capability (d) is in the process of being implemented in NAP). Two advanced features first introduced in NAP have been incorporated into SAGA, namely, solution of the normal equations by means of second order partitioned regression and consideration of serially correlated errors by autoregressive feedback. In Sections 4 and 5 of this report we shall provide the detailed development of the theories of partitioned regression and autoregressive feedback.

All four programs can recover coordinates of tracking stations. However, in forming the reduced system of normal equations, GDAP and NAP retain the system in core. This places a definite limit to the number of stations that can be solved for simultaneously (typically to 20 to 60 depending on computer). On the other hand, SAGA employs the logical development originally proved in MCT wherein the reduced normal equations are generated piecewise in core but are cumulatively formed on an external file (magnetic tape or disk). By this means it becomes practical to accommodate an overall tracking network embracing literally hundreds of unknown stations. The primary restriction is that only a limited subset of stations is regarded as participating successfully in the observation of any given pass (in SAGA the number is limited to a maximum of fifteen). Such a restriction is of no practical consequence in actual short arc operations, for rarely would as many as fifteen stations participate on a given pass, much less all be successful.

Error model coefficients appropriate to each channel of observations can be carried as adjustable constrained parameters in all four programs. In MCT and SAGA, all exercised error coefficients are regarded as unstable from pass to pass and thus are automatically reinitialized on each pass. GDAP and NAP also have this capability but are somewhat more flexible in that any desired subset of error coefficients can, on option, be treated as stable over a specified set of passes. For example, timing bias from a given station can be regarded, when desired, as stable over certain specified passes, rather than being reinitialized on each pass as are the other coefficients. This capability was not incorporated into SAGA primarily because it complicates considerably the logic and set up of the program and has proven to be a feature that is not often exercised in practice.

Of the four programs only GDAP has the option to perform geodetic reductions in a strictly geometric mode. This option was not incorporated into the other programs because experience with GDAP demonstrated the clear superiority in satellite geodesy of the short arc mode over the geometrical mode. In particular, recovery of tracking error coefficients has been found to be far superior in the short arc mode. Comparative analyses of the short arc versus the geometrical approach are made in Brown (1967a) and Brown (1968).

SAGA incorporates certain unique capabilities which experience with the other programs indicated would be desirable. One is consideration of random timing error. This was included primarily to make proper allowances in the reduction of PC-1000 chopping observations of passive satellites in view of studies indicating an rms mechanical jitter of about 0.3 ms in the operation of the chopping shutter. Inasmuch as relatively close satellite passes can cross the plate of a PC-1000 at rates up to 10mm/sec., an rms error in timing of 0.3 ms can be equivalent to as much as 3 microns on the plate and thus be comparable to plate measuring errors. Although it remains yet to be determined, random timing error might also potentially be of significance with the Geoceiver, particularly if rms noise levels in phase determination amounting to only about 0.1 m are achieved as projected in the design. Inasmuch as range rate of an observed satellite can amount to as much as 5000m/sec, an rms error in timing of as little as 20 microseconds can be equivalent to the expected rms error of 0.1 m in phase determination. Therefore, to be on the safe side, SAGA also makes provisions for the possibility of significant random timing error in electronic observations.

Another unique feature of SAGA is its ability to take rigorously into account errors in the adopted center of mass of the earth. As was pointed out in Brown (1967), when one elects in a short arc reduction to hold fixed the coordinates of a selected station, one thereby implicitly defines what is to be regarded as the location of the Earth's center of mass. The error in this implicitly adopted center of mass does have an effect (albeit a rather weak effect in most cases) on recovered coordinates of tracking stations. In the case of optical tracking networks of continental extent, the effect of such errors is very nearly equivalent to that of an error in scale. Through a series of exercises conducted with GDAP, Lynn, (1969) established, for example, that an error of 50 meters in the vertical component of the adopted location of the center of mass (vertical, that is, with respect to the fixed station) can cause an error in scale of about 1:200,000 in a continental network. In view of such findings, we incorporated into SAGA the capability of treating the coordinates of the adopted center of mass as constrained parameters. This means that in sufficiently strong tracking networks of continental extent, the possibility emerges of improving the location of the center of mass relative to the origin of the adopted datum. In weaker, more limited networks the main benefit of carrying coordinates of the center of mass as constrained parameters lies in the more comprehensive and realistic error propagation that is thereby produced (here, no significant improvement in the location of the center of mass is to be expected).

SAGA also differs from the other programs in that more comprehensive error models are employed for optical tracking systems. In addition SAGA is expressly designed to accommodate observations made by Geoceivers. Ranging error models that have so far been incorporated into GDAP and NAP are not sufficiently general to accommodate Geoceiver observations (should the need arise, however, they could readily be extended to do so). In the next two sections we shall go into the detailed development of the optical and electronic error models employed in SAGA.

From the foregoing review it can be appreciated that SAGA provides a powerful tool for satellite geodesy. What is not yet apparent is the fact that in spite of its flexibility SAGA has been designed to be easy to set up and use. This is accomplished in part by building into the program selectable sets of standard options sufficiently broad to cover most routine situations likely to be encountered in practice. Special situations can be accommodated when required, but at the expense of a more extensive set up of control parameters.

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2.0 OPTICAL OBSERVATIONAL EQUATIONS

2.1 Projective Equations

If, as in Figure 1, X,Y,Z denote the space coordinates of a point photographed by a camera located at X^c, Y^c, Z^c, it is well known that the image coordinates x,y are ideally given by the projective equations

$$
(1)\qquad
\begin{aligned}
x &= x_p + c\,\frac{A(X-X^c)+B(Y-Y^c)+C(Z-Z^c)}{D(X-X^c)+E(Y-Y^c)+F(Z-Z^c)}\\[2ex]
y &= y_p + c\,\frac{A'(X-X^c)+B'(Y-Y^c)+C'(Z-Z^c)}{D(X-X^c)+E(Y-Y^c)+F(Z-Z^c)}
\end{aligned}
$$

in which

x_p, y_p, c = coordinates of center of projection is image space (note: the z axis of image space is directed along the camera axis; thus c corresponds to focal length).

$\begin{bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{bmatrix}$ = orientation matrix = matrix of direction cosines of x,y,z of image space relative to X,Y,Z axes of object space (specifically, A,B,C are direction cosines of x relative to X,Y,Z; A'B'C' of y relative to X,Y,Z and D,E,F of z relative to X,Y,Z).

The orientation matrix can be expressed uniquely in terms of three angles. If the x,y,z axes of image space and the X,Y,Z axes of object space are related by the angles α, ω, κ indicated in Figure 2, the orientation matrix can be shown to assume the form:

$$
(2)\qquad
\begin{bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{bmatrix} =
\begin{bmatrix}
-\cos\alpha\cos\kappa - \sin\alpha\sin\omega\sin\kappa & \sin\alpha\cos\kappa - \cos\alpha\sin\omega\sin\kappa & \cos\omega\sin\kappa \\
\cos\alpha\sin\kappa - \sin\alpha\sin\omega\cos\kappa & -\sin\alpha\sin\kappa - \cos\alpha\sin\omega\cos\kappa & \cos\omega\cos\kappa \\
\sin\alpha\cos\omega & \cos\alpha\cos\omega & \sin\omega
\end{bmatrix},
$$

By dividing the numerator and denominator of the ratio on the right hand of (1) by:

$$
(3)\qquad R = [(X-X^c)^2 + (Y-Y^c)^2 + (Z-Z^c)^2]^{\frac{1}{2}}
$$

one can express the projective equations in terms of direction cosines:

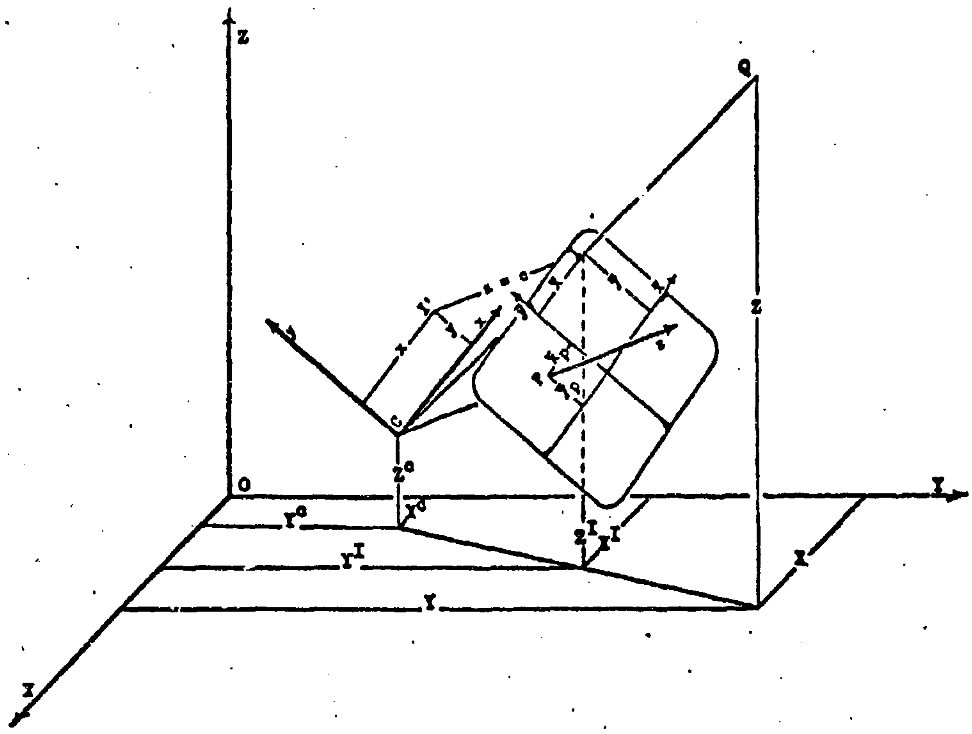

FIGURE 1

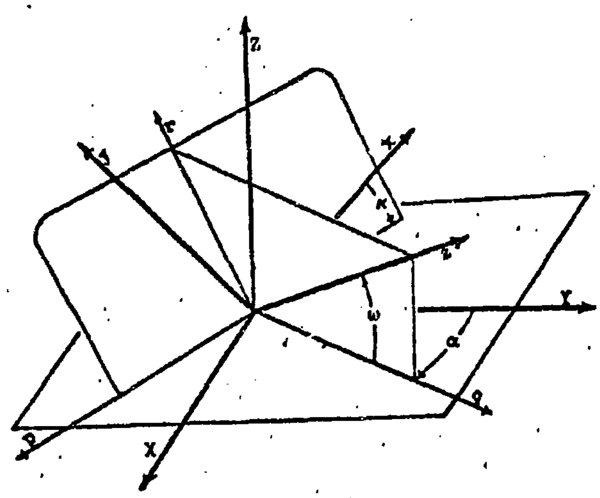

FIGURE 2

$$x = x_p + c \frac{A\lambda + B\mu + C\nu}{D\lambda + E\mu + F\nu}$$

(4)

$$y = y_p + c \frac{A'\lambda + B'\mu + C'\nu}{D\lambda + E\mu + F\nu}$$

wherein

(5) $\lambda = (X-X^c)/R, \quad \mu = (Y-Y^c)/R, \quad \nu = (Z-Z^c)/R.$

Equations (1) and (4) can be put into the alternate form:

$$\frac{\lambda}{\nu} = \frac{X-X^c}{Z-Z^c} = \frac{A(x-x_p) + A'(y-y_p) + Dc}{C(x-x_p) + C'(y-y_p) + Fc}$$

(6)

$$\frac{\mu}{\nu} = \frac{Y-Y^c}{Z-Z^c} = \frac{B(x-x_p) + B'(y-y_p) + Ec}{C(x-x_p) + C'(y-y_p) + Fc}.$$

The direction cosines λ, μ, ν can be expressed also as:

$$\lambda = \sin \alpha^* \cos \omega^*$$

(7) $\mu = \cos \alpha^* \cos \omega^*$

$$\nu = \sin \omega^*$$

in which α^*, ω^* are measured in the same sense as the α, ω that define the direction of the camera axis (Fig. 2).

Once the projective parameters $\alpha, \omega, x, x_p, y_p, c$ (and possibly others, such as coefficients of distortion) have been determined from a plate reduction based on measured plate coordinates of selected stars, the plate coordinates of satellite images can be employed in equations (6) and (7) to establish their directions α^*, ω^*. If X, Y, Z are suitably defined, α^*, ω^* will be equivalent to Greenwich hour angle and declination, which, in turn, can be converted into right ascension and declination if the time of the observation is known.

It has turned out that optical observations published by the various data gathering organizations consist of the derived quantities right ascension and declination (accompanied by time) instead of the original observations, namely, the measured plate coordinates. Thus the uncritical user is likely to regard right ascension and declination as directly observed quantities rather than as derived quantities. While errors in plate coordinates 'remain uncorrelated for all regions of the celestial sphere, errors in right ascension and declination become highly correlated in polar regions. Since most organizations do not accompany their published right ascensions and declinations with covariance matrices, such correlation is not generally taken properly into account. Consider, for example, the extreme case of a plate centered at the pole. If plate measuring accuracies are equal in x and y ($\sigma_x = \sigma_y = \sigma$), it can be shown that the standard deviations of the derived right ascension and declination (α, δ) are given by:

$$
(8) \qquad \begin{aligned} \sigma_\alpha &= (\sigma \tan \delta)/c \\ \sigma_\delta &= (\sigma \sin^2 \delta)/c \end{aligned}
$$

and the correlation between α, δ is given by:

$$
(9) \qquad \rho_{\alpha\delta} = 2 \sin \alpha \cos \alpha .
$$

Thus correlations between α, δ can range between -1 to +1 for points on the same plate.

To those versed in analytical photogrammetry, there is good reason to prefer plate coordinates over derived angles. Aside from the matter of correlation, the projective equations (1) relating x,y and X,Y,Z are actually simpler than the relations between α, δ and X,Y,Z. However, the overriding reason for preferring the projective equations has to do with error modeling. Systematic errors in optical directions are in large part attributable t[illegible]rors in the projective parameters produced by the plate re[illegible]tion. Especially significant, in many instances, is the angular instability of the camera throughout the data gathering period. As will shortly be demonstrated, a physically meaningful optical error model can be expressed in an especially compact form when the projective equations provide the observational equations for the reduction In particular, we shall show that four error coefficients can account for a total of eight distinct sources of systematic error.

Because plate coordinates and associated projective parameters are not generally available, one may resort to what may be termed the 'dummy camera method' to reconstruct from the given angles sets of plate coordinates that are approximately equivalent to those actually measured. The dummy camera method involves the following steps:

1) a focal length c is adopted that approximates the focal length of the camera actually used;

2) a central ray is selected to define the direction of the camera axis (α, ω) in equation (2);

3) with the swing angle $\varkappa$ provisionally set equal to zero, the orientation matrix of the dummy camera is evaluated from (2);

4) the given angles for each ray are converted into direction cosines λ, μ, ν by means of such relations as (7) (typically α^* corresponds to Greenwich hour angle computed from sidereal time and right ascension and ω^* corresponds to declination);

5) with x_p and y_p set to zero, with c equal to the value adopted in step (1) and with the orientation matrix computed in step (3), the direction cosines λ, μ, ν are substituted into eqs. (4) to generate equivalent plate coordinates x, y.

The dummy plate coordinates thus generated together with the adopted projective parameters of the dummy camera provide artifical observations having errors equivalent, for all practical purposes, to the errors in the original observations. For reasons shortly to be made clear, one extra step in the dummy camera projection is desirable. This is to redefine the swing angle $\varkappa$ (provisionally assigned a value of zero in step (3)) so that the x axis coincides approximately with the trace of the satellite. In this regard we would note that when the original observations are uncorrelated and are of the same accuracy ($\sigma_x = \sigma_y = \sigma$), so also are transformed values x', y' defined by:

$$x' = x \cos \varkappa - y \sin \varkappa$$

$$y' = y \sin \varkappa + x \cos \varkappa .$$

This means that the x axis of the dummy camera can be arbitrarily directed without significantly altering the error structure of the dummy observations. Accordingly, directing the x axis of the dummy camera along the traco of the satellite is altogether acceptable, even though this may not necessarily correspond to the direction of the original x axis.

2.2 Optical Error Model

The x, y coordinates reconstructed by the dummy camera method may be represented as:

$$x = x_p + c\,\frac{m}{q}$$

(10)

$$y = y_p + c\,\frac{n}{q}$$

in which

(11) $$\begin{bmatrix} m \\ n \\ q \end{bmatrix} = \begin{bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix} .$$

The systematic errors in x and y attributable to systematic errors in projective parameters may be represented as:

$$\delta x = \delta x_p + \frac{m}{q}\,\delta c + \frac{c}{q}\,\delta m - c\,\frac{m}{q^2}\,\delta q$$

(12)

$$\delta y = \delta y_p + \frac{n}{q}\,\delta c + \frac{c}{q}\,\delta n - c\,\frac{n}{q^2}\,\delta q.$$

The errors δm, δn, δq arise from errors in the orientation matrix. Let $\Delta\alpha, \Delta\omega, \Delta\kappa$ denote three infinitesimal rotations that serve to correct the orientation matrix. Then if m', n', q'

denote the correct values of m, n, q, one can write:

$$(13)\quad \begin{bmatrix} m' \\ n' \\ q' \end{bmatrix} = \begin{bmatrix} 1 & \Delta\varkappa & \Delta\alpha \\ -\Delta\varkappa & 1 & \Delta\omega \\ -\Delta\alpha & -\Delta\omega & 1 \end{bmatrix} \begin{bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix},$$

$$= \begin{bmatrix} 1 & \Delta\varkappa & \Delta\alpha \\ -\Delta\varkappa & 1 & \Delta\omega \\ -\Delta\alpha & -\Delta\omega & 1 \end{bmatrix} \begin{bmatrix} m \\ n \\ q \end{bmatrix}.$$

Here, it will be noted that the matrix in the quantities $\Delta\alpha, \Delta\omega, \Delta\varkappa$ qualifies as an orthogonal matrix if terms of second order are neglected when the matrix is multiplied by its transpose. Therefore, it is a rotation matrix. However, $\Delta\alpha, \Delta\omega, \Delta\varkappa$ do not consist of direct additive corrections to the $\alpha, \omega, \varkappa$ implicit in the original orientation matrix. Physically, $\Delta\alpha$ and $\Delta\omega$ are components of rotation of the camera axis in the xz and yz planes of image space, and $\Delta\varkappa$ is a component of rotation about the camera axis. The errors in m, n, q attributable to errors in the orientation matrix are given by:

$$(14)\quad \begin{bmatrix} dm \\ dn \\ dq \end{bmatrix} = \begin{bmatrix} m'-m \\ n'-n \\ q'-q \end{bmatrix} = \begin{bmatrix} 0 & \Delta\varkappa & \Delta\alpha \\ -\Delta\varkappa & 0 & \Delta\omega \\ -\Delta\alpha & -\Delta\omega & 0 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \\ q \end{bmatrix} = \begin{bmatrix} n\Delta\varkappa + q\Delta\alpha \\ -m\Delta\varkappa + q\Delta\omega \\ -m\Delta\alpha - n\Delta\omega \end{bmatrix}.$$

Before substituting these results into (12), we shall express $\Delta\alpha, \Delta\omega, \Delta\varkappa$ as:

$$(15)\quad \begin{aligned} \Delta\alpha &= \delta\alpha + \tau\ \delta\dot{\alpha} \\ \Delta\omega &= \delta\omega + \tau\ \delta\dot{\omega} \\ \Delta\varkappa &= \delta\varkappa + \tau\ \delta\dot{\varkappa} \end{aligned}$$

in which τ denotes the time of the observation relative to the time of the central ray defining the direction of the axis of the dummy camera. By virtue of (15), we adopt the assumption that the orientation of the camera is not necessarily strictly stationary

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but may be changing infinitesimally with time. If we now substitute (15) into (14) and then substitute this result into (12) we shall arrive at the expressions:

$$
\begin{aligned}
\delta x &= \delta x_p + \frac{x}{c}\,\delta c - c\left(1+\frac{x^2}{c^2}\right)\delta\alpha + \frac{xy}{c}\,\delta\omega + y\,\delta\kappa \\
&\qquad - c\left(1+\frac{x^2}{c^2}\right)\tau\,\delta\dot{\alpha} + \frac{xy}{c}\,\tau\,\delta\dot{\omega} + y\tau\,\delta\dot{\kappa}
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\delta y &= \delta y_p + \frac{y}{c}\,\delta c + \frac{xy}{c}\,\delta\alpha + c\left(1+\frac{y^2}{c^2}\right)\delta\omega - x\,\delta\kappa \\
&\qquad + \frac{xy}{c}\,\tau\,\delta\dot{\alpha} + c\left(1+\frac{y^2}{c^2}\right)\tau\,\delta\dot{\omega} - x\tau\,\delta\dot{\kappa}\,.
\end{aligned}
$$

In the reduction leading to this result we employed the relations $x = x(m/q)$, $y = c(n/q)$ which follow from the consideration that $x_p = y_p = 0$ in dummy camera projection.

As it stands, the error model (16) involves a total of nine parameters. However, the number can be reduced to a total of four essential parameters by certain considerations. First we note that for cameras of long focal length such as the MOTS and PC-1000, x and y are less than one tenth as great as c. Accordingly, terms x^2/c^2 and y^2/c^2 may be set equal to zero without significant effect. We now recall the fact that the swing angle κ is chosen in the adopted method of dummy camera projection so that the x axis coincides very nearly with the photographic trace of the satellite (which in turn typically departs from linearity by only a few hundred microns at most). Thus for all points on the trace $y \approx 0$. Moreover, the x coordinates of points along the trace can be represented approximately by the relation $x = \dot{x}\tau$ where $\dot{x}$ denotes the mean rate of change of x over the plate. If in line with these considerations we make the following set of substitutions into (16):

$$
x^2/c^2 = y^2/c^2 = 0, \quad y = 0, \; x = \dot{x}\tau \tag{17}
$$

we shall obtain the result:

$$(18a)\quad \delta x = \delta x_p - c\,\delta\alpha + \frac{\dot{x}\tau}{c}\,\delta c - c\tau\,\delta\dot{\alpha}$$

$$(18b)\quad \delta y = \delta y_p - c\,\delta\omega - \dot{x}\tau\,\delta\varkappa - c\tau\,\delta\dot{\omega} - \dot{x}t^2\,\delta\dot{\varkappa}.$$

We see in (18a) that the coefficients of δx_p and $\delta\alpha$ are constant multiples of each other; the same is true of the coefficients δc and $\delta\dot{\alpha}$. This means that $\delta\alpha$ alone is sufficient to account for the combined effects of infinitesimal changes in α and x_p. Likewise, δc alone is sufficient to account for the combined effects of an infinitesimal change in scale (or focal length) and an infinitesimal rate of change in the $\delta\alpha$ component of rotation. Similarly in (18b) we find that δy_p and $\delta\omega$ are perfectly coupled, as also are $\delta\varkappa$ and $\delta\dot{\omega}$. Thus, the rotations $\delta\omega$ and $\delta\varkappa$ serve also to account for δy_p and $\delta\dot{\omega}$ respectively. Further simplification can be achieved from consideration of the fact that with cameras having a focal length that is many times larger than the plate format, the term in $\delta\dot{\varkappa}$ is likely to be relatively insignificant in comparison with the terms in $\delta\dot{\alpha}$ and $\delta\dot{\omega}$. For cameras such as MOTS and PC-1000 the coefficients $c\tau$ of $\delta\dot{\alpha}$ and $\delta\dot{\omega}$ in (18a) and (18b) are about ten times larger than the maximum value of the coefficient $\dot{x}t^2 = x\tau$ of $\delta\dot{\varkappa}$. This means that $\delta\dot{\varkappa}$ must be about ten times greater than $\delta\dot{\alpha}$ and $\delta\dot{\omega}$ in order to induce a comparable error. In a study of camera stability reported by Brown (1969) the maximum values of $\delta\dot{\alpha}, \delta\dot{\omega}$, and $\delta\dot{\varkappa}$ for a PC-1000 were found to be about $0''.01$/sec for $\delta\dot{\alpha}$ and $\delta\dot{\omega}$ and about $0''.02$/sec for $\delta\dot{\varkappa}$. Although $\delta\dot{\varkappa}$ did become about twice as great as $\delta\dot{\alpha}, \delta\dot{\omega}$ its net effect was only one fifth as great inasmuch as $|\max x| \approx 0.1\,c$. In view of such considerations, we regard carrying $\delta\dot{\varkappa}$ in the error model to be generally of dubious value and accordingly have dropped it in further treatment of the model.

By virtue of the findings of the previous paragraph, we may drop from the general error model (16) the terms in δx_p $\delta y_p, \delta\dot{\alpha}, \delta\dot{\omega}$ and $\delta\dot{\varkappa}$. This leaves a four parameter model of the form:

$$(19)\quad \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} -c\left(1 + \dfrac{x^2}{c^2}\right) & \dfrac{xy}{c} & y & \dfrac{x}{c} \\ \dfrac{xy}{c} & c\left(1 + \dfrac{y^2}{c^2}\right) & -x & \dfrac{y}{c} \end{bmatrix} \begin{bmatrix} \delta\alpha \\ \delta\omega \\ \delta\varkappa \\ \delta c \end{bmatrix}.$$

This compact model is sufficient to account not only for biases in six projective parameters but also for my uniform drift of the camera axis throughout the exposure. One must not lose sight of the fact that this result does depend in part on dummy camera projection that places the trace of the satellite through the plate center and approximately along the x axis of the plate.

When optical systems are employed to record a flashing light, synchronization of all observations is automatic and the problem of interstation timing bias does not arise. However, when shutters are employed to chop the traces of sun illuminated passive satellites, the possibility does arise that local clocks may be inadequately synchronized. In this case the error model (19) must be augmented by terms of the form:

$$
\begin{aligned}
\delta x_t &= \dot{x}\,\delta t \\
\delta y_t &= \dot{y}\,\delta t,
\end{aligned}
\tag{20}
$$

where δt represents the interstation timing bias. In a short arc tracking network δt can and should be forced equal to zero for one arbitrarily selected station in the network. The biases δt for the remaining stations are subject to a priori constraints appropriate to the timing system employed.

In cases where optical and electronic systems both track a satellite carrying a flashing light, interstation timing bias is accommodated in SAGA by treating the timing of the optical system as unbiased and the timing of the electronic systems as biased relative to the optical system.

A common and desirable practice in optical tracking is to reorient each camera one or more times during the course of a pass in order to obtain extended coverage from a given station. MOTS cameras occasionally obtain as many as four plates on a given pass and two or three plates are common. Under such circumstances it becomes necessary to reinitialize the error model for each plate (except for interstation timing bias which would be common to all plates at a given station for a given pass). This means that if a particular station were to acquire four plates on a given pass, one would have to determine an independent set of coefficients $\delta\alpha, \delta\omega, \delta\kappa, \delta c$ for each plate and, where applicable, a single interstation timing bias δt for all plates. As a consequence, an optical station can require the exercise of as many as seventeen error coefficients for a single pass. Such a capability is provided in SAGA. Let us consider what this implies in view of the fact that SAGA is designed to accommodate as many as fifteen stations on a given pass. The most extreme situation would be one in which all fifteen stations are employed in a chopping mode and each station successfully acquires four plates. The number of error parameters to be recovered on a single pass would then amount to $15 \times 17 - 1 = 254$ (the timing bias at one station is constrained to zero). Such a reduction becomes practical only by virtue of the use of second order partitioned regression as is discussed in Section 4.

2.3 A Priori Constraints

By virtue of the stellar control employed in plate reductions, systematic errors in [illegible]ly determined directions are sharply bounded. The error budget for a PC-1000 reduction provided in Table 2 is taken from Brown, Bush, Sibol (1963). For current validity the budget need be changed in only a few respects. The use of the SAO star catalog in place of the Boss catalog would about halve the contribution of item A3. Tangential or lens decentering distortion is now routinely calibrated and removed according to methods developed in Brown (1964), (1966). As a result items A6 and B6 of the error budget can be reduced to about one third their former values. The most significant change to the budget affects item A11 which is concerned with camera stability. The budget calls for rejection of the plate if comparison between pre and post orientations indicates the presence of camera instability equivalent to more than one third the net rms error in the plate coordinates. This recommendation has been found to be too stringent to be followed in general practice. Instead, instability is tolerated to the point where its effects on direction are comparable with those random errors. In effect, this means that some PC-1000 plates are accepted even though a change in orientation of as much as two seconds of arc exists between pre and post calibrations. When such a change is continuous (as opposed to a sudden disturbance), its effect on

TABLE 2 ERROR BUDGET FOR PC-1000 FOR POINTS OUTSIDE ATMOSPHERE

	ERROR SOURCE	RMS CONTRIBUTION IN MICRONS UNDER: (a) Favorable Conditions	(b) Normal Conditions	(c) Unfavorable Conditions
A. CALIBRATION OF ORIENTATION	1. Random setting error (average of 2 settings).	1.0	1.5	2.0
	2. Emulsion instability.	1.0	1.5	2.5
	3. Low frequency atmospheric shimmer.	0.5	1.0	3.0
	4. Star catalog error (Boss).	2.0	3.0	4.0
	5. Residual radial distortion.	0.5	1.0	2.0
	6. Residual tangential distortion.	1.0	2.0	4.0
	7. Flatness of surface of emulsion.	0.0	0.0	0.5
	8. Residual differential refraction	0.0	0.5	1.0
	9. Residual comparator errors.	0.5	1.0	1.5
	10. Timing errors (WWV).	0.5	1.0	2.0
	11. Camera instability (below threshold of routine detectability).	1.0	1.5	2.0
	POOLED RSS TOTALS:	3.0 μ	4.9 μ	8.2 μ
B. DETERMINATION OF DIRECTIONS OF FLASHES	1. Random setting error (average of 2 settings).	1.0	1.5	2.0
	2. Emulsion instability.	1.0	1.5	2.5
	3. High frequency atmospheric shimmer.	1.0	2.5	5.0
	4. Residual error in calibrated orientation.	0.5	1.0	1.5
	5. Residual radial distortion.	1.0	1.5	2.5
	6. Residual tangential distortion.	1.0	2.0	4.0
	7. Flatness of surface of emulsion.	0.0	0.0	0.5
	8. Residual parallactic refraction.	0.5	1.0	1.5
	9. Residual comparator errors.	0.5	1.0	1.5
	POOLED RSS TOTALS:	2.4 μ	4.5 μ	8.0 μ

GENERAL QUALIFICATIONS:

a. Calibration is assumed to involve at least 40 stellar images compactly distributed about flashing light trace and divided approximately equally between pre- and postcalibrations.

b. Elevation angle of camera is taken as 30° and altitude of flashes as 400 nm.

c. Photoprocessing procedure recommended by Gallnow and Hageman (Astronomical Journal, pp. 399-404, Vol. 61, Nov. 1956) is assumed employed in order to minimize emulsion instability; for same reason, points within one centimeter of edge of plate are assumed not to be measured.

d. Atmospheric shimmer is taken to be that characteristic of maritime subtropical atmosphere at 30° elevation angle with PC-1000 employed at full (200 mm) aperture.

e. Timing errors (WWV) are taken as 5, 10, and 20 milliseconds for cases (a), (b), (c), respectively.

f. Comparator is assumed to be calibrated and properly operated.

g. Plates are assumed [illegible] rejected if comparison between individual pre and post orientations indicate presence of cam[illegible] bility equivalent to more than one third the net rms error in the plate coordinates.

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satellite directions is likely to be less than one second of arc because of the relatively short time interval spanned by the satellite observations.

In view of current practice, we would suggest that a priori constraints for PC-1000 and MOTS optical error coefficients be selected from one of the following three schedules:

Schedule	Criterion	σ_α	σ_ω	σ_χ	σc
1 (favorable)	$\sigma_s < 3\mu$	0".5	0".5	1".0	10μ
2 (normal)	$3 < \sigma_s < 6\mu$	1".0	1".0	2".0	20μ
3 (unfavorable)	$\sigma_s > 6\mu$	1".5	1".5	3".0	30μ

The quantity σ_s refers to the rms error achieved in the plate reduction.

A primary advantage of the short arc approach to satellite geodesy over the geometric approach is the practibility of accounting for systematic errors in extensive networks through error modeling. On strongly observed arcs, adjusted values of many of the error coefficients can constitute substantial improvements over a priori values. On the other hand, some error coefficients may prove to be intractible, their accuracies after adjustment being no better than before adjustment. This has been used as an argument against the exercise of error models in the adjustment. Such an argument is unsound for it is clearly important that the effects of statistically bounded systematic errors be rigorously taken into account even when such errors are not amenable to worthwhile reduction. This is especially so in the case of plates containing a large number of satellite images, as when passive satellites are recorded. Here, one would obtain unduly optimistic estimates of accuracy from error propagation if one were to ignore the possible existence of systematic error.

2.4 Special Corrections

SAGA is designed to accept any optical observations that are provided in the GEOS format of the NASA data bank. Unfortunately, there is no uniform standard with regard to certain corrections that is followed by all agencies producing optical data. In particular, the following corrections may or may not have been applied by a given agency:

1) polar motion;
2) conversion of times to UT1;
3) parallactic refraction;
4) phase angle correction (chopping of passive satellites);
5) propagation delay (chopping of passive satellites).

Because of the lack of homogeneity in the application of such corrections, SAGA has been provided with a special preprocessor which serves to apply these corrections as needed. Characteristics of the preprocessor are discussed in Part II of this report.

2.5 Orbital Constraints

The orbital integrator employed by SAGA is that developed by Hartwell (1968), (1969). It employs a power series solution to the equations of motion wherein each coefficient is formed in terms of its predecessors by means of recursive algorithms. The version of the integrator employed in SAGA truncates the gravitational potential at (n,m) = (4,4) inasmuch as this has proven to be entirely adequate in short arc applications (Brown 1967). If x,y,z denote the geocentric inertial coordinates at an arbitrary time τ relative to an adopted epoch $\tau=0$, the power series solution of the equations of motion can be represented as:

$$(21) \quad \begin{bmatrix} x & \dot{x} \\ y & \dot{y} \\ z & \dot{z} \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & a_2 \dots a_p \\ b_0 & b_1 & b_2 \dots b_p \\ c_0 & c_1 & c_2 \dots c_p \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \tau & 1 \\ \tau^2 & 2\tau \\ \vdots & \vdots \\ \tau^p & p\tau^{p-1} \end{bmatrix}$$

in which all of the coefficients are functions of the six initial conditions at $\tau = 0$

(namely: $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0$) and gravitational coefficients. The series is truncated automatically when a prespecified tolerance (presently taken as 0.001 m) is satisfied for the maximum value of τ to be exercised. If the epoch is taken near midarc, the radius of convergence of each expansion is sufficiently great to accommodate arcs as long as one third of a revolution for nearly circular orbits.

The version of the integrator employed in SAGA also generates power series solutions to the variational equations relating errors in x, y, z at time τ to errors in the adopted location of the center of mass and errors in the six initial conditions. If we let X_{00}, Y_{00}, Z_{00} denote the earth-fixed coordinates of the center of mass, the matrix of partial derivatives generated by the integrator (the matrizant) can be expressed as

$$(22)\quad \underset{(3,9)}{\Omega} = \frac{\partial(x,y,z;t)}{\partial(X_{00}, Y_{00}, Z_{00}, x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{19} \\ \Omega_{21} & \Omega_{22} & \cdots & \Omega_{29} \\ \Omega_{31} & \Omega_{32} & \cdots & \Omega_{39} \end{bmatrix}$$

in which each $\Omega_{\ell m}$ is, in turn, a polynomial:

$$(23)\quad \Omega_{\ell m} = (\alpha_0 \;\; \alpha_1 \;\; \alpha_2 \; \cdots \; \alpha_q)_{\ell m} \begin{bmatrix} 1 \\ \tau \\ \tau^2 \\ \vdots \\ \tau^q \end{bmatrix}.$$

Inertial coordinates generated by the orbital integrator can be referred to an earth-fixed framework by the application of the transformation:

$$(24)\quad \begin{bmatrix} X \\ Y \\ Z \\ \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} R & 0 \\ \dot{R} & R \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

in which

$$
(25) \qquad R = \begin{bmatrix} \cos\psi\tau & \sin\psi\tau & 0 \\ -\sin\psi\tau & \cos\psi\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \dot{R} = \psi \begin{bmatrix} -\sin\psi\tau & \cos\psi\tau & 0 \\ -\cos\psi\tau & -\sin\psi\tau & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

In a similar manner the matrizant can be transformed to earth fixed coordinates by the operation:

$$
(26) \qquad \underset{(3,9)}{\Phi^{\cdot}} = \underset{(3,3)}{R}\ \underset{(3,9)}{\Omega} = \frac{\partial(X, Y, Z; \tau)}{\partial(X_{\infty}, Y_{\infty}, Z_{\infty}, x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)}.
$$

With these results we are now in a position to develop the optical observational equations for the short arc reduction.

2.6 Optical Observational Equations

If the X, Y, Z in the projective equations (1) are replaced by the values computed from (24), the equations become functionally of the form:

$$
(27) \qquad \begin{aligned} x &= f_1 (X^c, Y^c, Z^c, X_{\infty}, Y_{\infty}, Z_{\infty}, x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0; t) \\ y &= f_2 (X^c, Y^c, Z^c, X_{\infty}, Y_{\infty}, Z_{\infty}, x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0; t). \end{aligned}
$$

If x^0, y^0 denote the observed values of x, y, the adjusted values corrected for systematic error can be expressed as:

$$
(28) \qquad \begin{aligned} x &= x^0 + v_x + \dot{x} v_t + \dot{x}\,\delta t + \delta x \\ y &= y^0 + v_y + \dot{y} v_t + \dot{y}\,\delta t + \delta y \end{aligned}
$$

where $\delta x, \delta y$ are given by (19). In (28) v_x, v_y denote residuals reflecting random error in x and y, and v_t denotes the residual reflecting random error timing. The terms in δt account for the bias in timing (eq. (20)). We now set up the relations:

$$
(29)\qquad
\begin{array}{llll}
X^c = (X^c)^{oo} + \delta X^c & X_\infty = X_\infty^{oo} + \delta X_\infty & x_o = x_o^{oo} + \delta x_o & \dot{x}_o = \dot{x}_o^{oo} + \delta \dot{x}_o \\
Y^c = (Y^c)^{oo} + \delta Y^c & Y_\infty = Y_\infty^{oo} + \delta Y_\infty & y_o = y_o^{oo} + \delta y_o & \dot{y}_o = \dot{y}_o^{oo} + \delta \dot{y}_o \\
Z^c = (Z^c)^{oo} + \delta Z^c & Z_\infty = Z_\infty^{oo} + \delta Z_\infty & z_o = z_o^{oo} + \delta z_o & \dot{z}_o = \dot{z}_o^{oo} + \delta \dot{z}_o
\end{array}
$$

in which the superscripts (oo) denote approximations and the δ's are corresponding corrections. Substituting these into the right hand side of (27) and linearizing the resulting expressions by Taylors series, we obtain:

$$
(30)\qquad
\begin{aligned}
x &= x^{oo} + \frac{\partial x}{\partial(X^c, Y^c, Z^c, X_\infty, Y_\infty, Z_\infty, x_o, y_o, z_o, \dot{x}_o, \dot{y}_o, \dot{z}_o)} (\delta X^c, \delta Y^c, \delta Z^c, \ldots, \delta\dot{y}_o, \delta\dot{z}_o)^T \\
y &= y^{oo} + \frac{\partial y}{\partial(X^c, Y^c, Z^c, X_\infty, Y_\infty, Z_\infty, x_o, y_o, z_o, \dot{x}_o, \dot{y}_o, \dot{z}_o)} (\delta X^c, \delta Y^c, \delta Z^c, \ldots, \delta\dot{y}_o, \delta\dot{z}_o)^T
\end{aligned}
$$

in which

$$
(31)\qquad
\begin{aligned}
x^{oo} &= f_1((X^c)^{oo}, (Y^c)^{oo}, \ldots, \dot{z}_o^{oo}) \\
y^{oo} &= f_2((X^c)^{oo}, (Y^c)^{oo}, \ldots, \dot{z}_o^{oo}).
\end{aligned}
$$

To arrive at explicit expressions for the elements of the linearized observational equations, we let $X^{oo}, Y^{oo}, Z^{oo}, \dot{X}^{oo}, \dot{Y}^{oo}, \dot{Z}^{oo}$ denote the components of position and velocity for the time τ of the observation as computed from equations (21) and (24) in which the given approximations in (29) are employed in the integration. Then we define the auxiliary:

$$
(32)\qquad
\begin{bmatrix} m^{oo} \\ n^{oo} \\ q^{oo} \end{bmatrix}
=
\begin{bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{bmatrix}
\begin{bmatrix} X^{oo} - (X^c)^{oo} \\ Y^{oo} - (Y^c)^{oo} \\ Z^{oo} - (Z^c)^{oo} \end{bmatrix}
$$

where the orientation matrix has been developed by dummy camera projection. The values of the plate coordinates computed from (31) then become:

$$(33)\quad \begin{bmatrix} x^{\infty} \\ y^{\infty} \end{bmatrix} = \frac{c}{q^{\infty}} \begin{bmatrix} m^{\infty} \\ n^{\infty} \end{bmatrix}$$

where c denotes the focal length adopted in dummy camera projection. The partial derivatives of the plate coordinates with respect to station coordinates is given by:

$$(34)\quad \underset{(2,3)}{B^{(1)}} = \frac{\partial(x,y)}{\partial(X^c,Y^c,Z^c)} = \frac{-c}{q^{\infty}} \begin{bmatrix} 1 & 0 & -x^{\infty}/c \\ 0 & 1 & -y^{\infty}/c \end{bmatrix} \begin{bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{bmatrix}.$$

In terms of this the partial derivatives of the plate coordinates with respect to center of mass and orbital state vector are given by:

$$(35)\quad \underset{(2,9)}{B^{(2)}} = \frac{\partial(x,y)}{\partial(X_{\infty}, Y_{\infty}, \ldots, \dot{Z}_0)} = -\underset{(2,3)}{B^{(1)}} \; \underset{(3,9)}{\Phi}$$

in which Φ is given by (26). The time derivatives of the plate coordinates required in (28) can be computed from:

$$(36)\quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underset{(2,3)}{B^{(1)}} \begin{bmatrix} \dot{X}^{\infty} \\ \dot{Y}^{\infty} \\ \dot{Z}^{\infty} \end{bmatrix}.$$

If we partition $B^{(2)}$ as:

$$(37)\quad \underset{(2,9)}{B^{(2)}} = \left[\frac{\partial(x,y)}{\partial(X_{\infty}, Y_{\infty}, Z_{\infty})} \;\vdots\; \frac{\partial(x,y)}{\partial(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)} \right] = \left[\underset{(2,3)}{B^{(2a)}} \;\vdots\; \underset{(2,6)}{B^{(2b)}} \right]$$

the linearized observation equations can be put into the form:

$$(38)\quad \underset{(2,3)(3,1)}{A \quad v} - \underset{(2,3)(3,1)}{B^{(1)} \; \delta^{(1)}} - \underset{(2,3)(3,1)}{B^{(2a)} \; \delta^{(2a)}} - \underset{(2,6)(6,1)}{B^{(2b)} \; \delta^{(2b)}} - \underset{(2,1)(1,1)}{B^{(3)} \; \delta^{(3)}} - \underset{(2,4)(4,1)}{B^{(4)} \; \delta^{(4)}} = \underset{(2,1)}{\epsilon}$$

in which

$$A = \begin{bmatrix} 1 & 0 & \dot{x} \\ 0 & 1 & \dot{y} \end{bmatrix}, \quad v = \begin{bmatrix} v_x \\ v_y \\ v_t \end{bmatrix}, \quad \delta^{(1)} = \begin{bmatrix} \delta X^c \\ \delta Y^c \\ \delta Z^c \end{bmatrix}, \quad \delta^{(2a)} = \begin{bmatrix} \delta X_{\infty} \\ \delta Y_{\infty} \\ \delta Z_{\infty} \end{bmatrix}, \quad \delta^{(2b)} = \begin{bmatrix} \delta x_0 \\ \delta y_0 \\ \vdots \\ \delta \dot{z}_0 \end{bmatrix}$$

(39) $$B^{(3)} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}, \quad \delta^{(3)} = (\delta t),$$

$$B^{(4)} = -\begin{bmatrix} -c\left(1 + \frac{x^2}{c^2}\right) & \frac{xy}{c} & y & \frac{x}{c} \\ \frac{xy}{c} & c\left(1 + \frac{y^2}{c^2}\right) & -x & \frac{y}{c} \end{bmatrix}, \quad \delta^{(4)} = \begin{bmatrix} \delta\alpha \\ \delta\omega \\ \delta\kappa \\ \partial c \end{bmatrix}.$$

At this point we shall recognize that as many as four plates may be acquired at a given station for a given pass. Accordingly, there may be as many as four sets of error coefficients. Letting p denote the pth plate (max p=4) recorded and introducing the subscript j to denote the jth point observed by the station, we may express the pair of linearized observational equations generated jth point, if observed on plate p, as:

(40) $$A_j v_j + B_j \delta = \epsilon_j$$

in which

$$
\begin{array}{llcccccccc}
 & & & \text{Station Coord.} & \text{Center of Mass Coord.} & \text{Orbital State Vector} & \text{Timing Bias} & \text{Error Coeff., Plate 1} & \text{Error Coeff., Plate 2} & \text{Error Coeff., Plate 3} & \text{Error Coeff., Plate 4} \\
(41) & \underset{(2,12+\ell)}{B_j} & = & -[\underset{(2,3)}{B^{(1)}_{pj}} & \underset{(2,3)}{B^{(2a)}_{pj}} & \underset{(2,6)}{B^{(2b)}_{pj}} & \underset{(2,1)}{B^{(3)}_{pj}} & \underset{(2,4)}{\xi_{1p} B^{(4)}_{pj}} & \underset{(2,4)}{\xi_{2p} B^{(4)}_{pj}} & \underset{(2,4)}{\xi_{3p} B^{(4)}_{pj}} & \underset{(2,4)}{\xi_{4p} B^{(4)}_{pj}}]
\end{array}
$$

$$
(42) \qquad \underset{(12+\ell,1)}{\delta} = (\underset{(1,3)}{\delta^{(1)^T}} \; \underset{(1,3)}{\delta^{(2a)^T}} \; \underset{(1,6)}{\delta^{(2b)^T}} \; \underset{(1,1)}{\delta^{(3)^T}} \; \underset{(1,4)}{\delta^{(4)^T}} \; \underset{(1,4)}{\delta^{(4)^T}} \; \underset{(1,4)}{\delta^{(4)^T}} \; \underset{(1,4)}{\delta^{(4)^T}})^T
$$

The dimension ℓ in these expressions is introduced to denote the total number of error coefficients exercised by the given station on the given pass. The quantity ξ_{ip} is defined as:

$$
(43) \qquad \begin{aligned} \xi_{ip} &= 1 \text{ if } i=p \\ \xi_{ip} &= 0 \text{ if } i \neq p. \end{aligned}
$$

In this formulation it is understood that the number of parameters generated by a given station for a given pass increases by four with each plate successfully recorded. Thus B_j may range from a minimum of a (2,17) matrix for a single plate to a maximum of a (2,29) for a set of four plates.

2.7 Normal Equations

We are now in a position to consider the formation of the normal equations for optical observations. In doing so, we shall employ the methodology employed in Brown, Trotter (1967). Accordingly, we first form the normal equations for a given station and pass, ignoring the existence of other stations and other passes. If the covariance matrix of the random errors in plate coordinates and timing for the j th point from the given station is denoted by:

$$(44) \qquad \underset{(3,3)}{\Lambda_j} \quad \begin{bmatrix} \sigma^2_{x_j} & 0 & 0 \\ 0 & \sigma^2_{y_j} & 0 \\ 0 & 0 & \sigma^2_{T_j} \end{bmatrix},$$

the system of normal equations generated by the point can be expressed as:

$$(45) \qquad N_j \, \delta = c_j$$

in which

$$(46) \qquad \begin{aligned} N_j &= B_j^T (A_j \Lambda_j A_j^T)^{-1} B_j \\ c_j &= B_j^T (A_j \Lambda_j A_j^T)^{-1} \epsilon_j . \end{aligned}$$

It is to be noted that since:

$$(47) \qquad A_j \Lambda_j A_j^T = \begin{bmatrix} \sigma^2_{x_j} + \dot{x}_j^2 \sigma_{T_j}^2 & 0 \\ 0 & \sigma^2_{y_j} + \dot{y}_j^2 \sigma^2_{T_j} \end{bmatrix}$$

the results of the multiplication by $(A_j \Lambda_j A_j)^{-1}$ can also be effected by treating this matrix as a unit matrix in (46) after modifying B_j and ϵ_j by dividing their first and second rows, respectively, by $(\sigma^2_{x_j} + \dot{x}_j^2 \sigma^2_{T_j})^{\frac{1}{2}}$ and $(\sigma^2_{y_j} + \dot{y}_j^2 \sigma^2_{T_j})^{\frac{1}{2}}$.

The system normal equations generated by all points from the given station and pass is simply:

$$(48) \qquad N \, \delta = c$$

where

$$(48a) \qquad \begin{aligned} N &= \Sigma N_j \\ c &= \Sigma c_j . \end{aligned}$$

We now introduce subscripts i and k to denote the i th station and k th pass respectively. The normal equations (48) may then be written in more detailed partitioned form as:

$$
(49)\qquad
\begin{array}{c}
\begin{array}{cccc}
\text{Coord., Station } i & \text{Center of Mass} & \text{State Vector, Pass } k & \text{Error Coeff., Sta. } i\text{, Pass } k
\end{array}\\
\begin{bmatrix}
\underset{(3,3)}{[\hat{U}_{11}]_{ik}} & \underset{(3,3)}{[\hat{U}_{12}]_{ik}} & \underset{(3,6)}{[\dot{U}_{1}]_{ik}} & \underset{(3,\ell_{ik})}{[\bar{U}_{1}]_{ik}} \\
\underset{(3,3)}{[\hat{U}_{12}]^T_{ik}} & \underset{(3,3)}{[\hat{U}_{22}]_{ik}} & \underset{(3,6)}{[\dot{U}_{2}]_{ik}} & \underset{(3,\ell_{ik})}{[\bar{U}_{2}]_{ik}} \\
\underset{(6,3)}{[\dot{U}_{1}]^T_{ik}} & \underset{(6,3)}{[\dot{U}_{2}]^T_{ik}} & \underset{(6,6)}{\dot{N}_{ik}} & \underset{(6,\ell_{ik})}{\bar{N}_{ik}} \\
\underset{(\ell_{ik},3)}{[\bar{U}_{1}]^T_{ik}} & \underset{(\ell_{ik},3)}{[\bar{U}_{2}]^T_{ik}} & \underset{(6,\ell_{ik})}{\bar{N}^T_{ik}} & \underset{(\ell_{ik},\ell_{ik})}{\ddot{N}_{ik}}
\end{bmatrix}
\end{array}
\begin{bmatrix}
\underset{(3,1)}{\hat{\delta}_i} \\
\underset{(3,1)}{\hat{\delta}_{00}} \\
\underset{(6,1)}{\dot{\delta}_k} \\
\underset{(\ell_{ik},1)}{\ddot{\delta}_{ik}}
\end{bmatrix}
=
\begin{bmatrix}
\underset{(3,1)}{[\hat{c}_1]_{ik}} \\
\underset{(3,1)}{[\hat{c}_2]_{ik}} \\
\underset{(6,1)}{\dot{c}_{ik}} \\
\underset{(\ell_{ik},1)}{\ddot{c}_{ik}}
\end{bmatrix}.
$$

We shall find it convenient to proceed formally as if all m stations in the tracking network were to observe all passes (presently, this assumption will be dropped). If we then assume that (49) has been evaluated for all m stations ($i = 1, 2, \ldots, m$), and merge the resulting individual sets of normal equations into a common system by the process of zero augmentation developed in Brown, Trotter (1967), we shall obtain the system (50) indicated on the next page. If a particular station does not participate in the tracking of the k th pass, equation (50) should be modified by (a) replacing all elements corresponding to the station in the $\hat{U}$ and $\hat{c}$ portions of the normal equations by zeroes, and (b) deleting from the remainder of the normal equations the elements corresponding to the station.

By adopting the partitioning indicated by the broken lines in (50), we can represent the system of normal equations for the k th pass in the more compact form:

$$
\begin{array}{ccccc|c|cccc}
\multicolumn{4}{c}{\text{Coord. of Tracking Stations}} & & & \multicolumn{4}{c}{\text{Error Model Coefficients}} \\
\text{Station 1} & \text{Station 2} & & \text{Station m} & \text{Center of Mass} & \text{State Vector, Pass k} & \text{Station 1} & \text{Station 2} & & \text{Station m}
\end{array}
$$

$$
\left[\begin{array}{ccccc|c|cccc}
[\hat{U}_{11}]_{1k} & 0 & \cdots & 0 & [\hat{U}_{12}]_{1k} & [\dot{U}_1]_{1k} & [\bar{U}_1]_{1k} & 0 & \cdots & 0 \\
0 & [\hat{U}_{11}]_{2k} & \cdots & 0 & [\hat{U}_{12}]_{2k} & [\dot{U}_1]_{2k} & 0 & [\bar{U}_1]_{2k} & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & [\hat{U}_{11}]_{mk} & [\hat{U}_{12}]_{mk} & [\dot{U}_1]_{mk} & 0 & 0 & \cdots & [\bar{U}_1]_{mk} \\
[\hat{U}_{12}]^T_{1k} & [\hat{U}_{12}]^T_{2k} & \cdots & [\hat{U}_{12}]^T_{mk} & \sum_{i=1}^{m}[\hat{U}_{22}]_{ik} & \sum_{i=1}^{m}[\dot{U}_2]_{ik} & [\bar{U}_2]_{1k} & [\bar{U}_2]_{2k} & \cdots & [\bar{U}_2]_{mk} \\
\hline
[\dot{U}_1]^T_{1k} & [\dot{U}_1]^T_{2k} & \cdots & [\dot{U}_1]^T_{mk} & \sum_{i=1}^{m}[\dot{U}_2]^T_{ik} & \sum_{i=1}^{m}\dot{N}_{ik} & \bar{N}_{1k} & \bar{N}_{2k} & \cdots & \bar{N}_{mk} \\
\hline
[\bar{U}_1]^T_{1k} & 0 & \cdots & 0 & [\bar{U}_2]^T_{1k} & \bar{N}_{1k} & \ddot{N}_{1k} & 0 & \cdots & 0 \\
0 & [\bar{U}_1]^T_{2k} & \cdots & 0 & [\bar{U}_2]^T_{2k} & \bar{N}_{2k} & 0 & \ddot{N}_{2k} & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & [\bar{U}_1]^T_{mk} & [\bar{U}_2]^T_{mk} & \bar{N}_{mk} & 0 & 0 & \cdots & \ddot{N}_{mk}
\end{array}\right]
\left[\begin{array}{c}
\hat{\delta}_1 \\ \hat{\delta}_2 \\ \vdots \\ \hat{\delta}_m \\ \hat{\delta}_{00} \\ \hline \dot{\delta}_k \\ \hline \ddot{\delta}_{1k} \\ \ddot{\delta}_{2k} \\ \vdots \\ \ddot{\delta}_{mk}
\end{array}\right]
=
\left[\begin{array}{c}
[\hat{c}_1]_{1k} \\ [\hat{c}_1]_{2k} \\ \vdots \\ [\hat{c}_1]_{mk} \\ \sum_{i=1}^{m}[\hat{c}_2]_{ik} \\ \hline \sum_{i=1}^{m}\dot{N}_{ik} \\ \hline \ddot{c}_{1k} \\ \ddot{c}_{2k} \\ \vdots \\ \ddot{c}_{mk}
\end{array}\right]
\tag{50}
$$

$$
(51)\qquad
\begin{array}{c}
\begin{array}{ccc}
\text{Coord. all Stations and Center of Mass} & \text{State Vector, Pass k} & \text{Error Coeff., all Stations, Pass k}
\end{array}\\
\begin{bmatrix}
\hat{U}_k & \dot{U}_k & \bar{U}_k \\
\dot{U}_k^T & \dot{N}_k & \bar{N}_k \\
\bar{U}_k^T & \bar{N}_k^T & \ddot{N}_k
\end{bmatrix}
\end{array}
\begin{bmatrix}
\hat{\delta} \\ \dot{\delta}_k \\ \ddot{\delta}_k
\end{bmatrix}
=
\begin{bmatrix}
\hat{c} \\ \dot{c}_k \\ \ddot{c}_k
\end{bmatrix}.
$$

We next assume that (51) has been evaluated for all n observed passes $(k = 1, 2, \ldots, n)$ and again resort to the process of zero augmentation to merge all such individual sets of normal equations into a common system. This leads to the system indicated in equation (52) below. We shall assume that weight matrices $\hat{W}$, $\dot{W}_k$, $\ddot{W}_k$ reflecting the a priori constraints to be exercised in the adjustment have been absorbed into the appropriate diagonal blocks of (52). Then (52) represents the final system of normal equations.

$$
(52)\qquad
\begin{array}{c}
\begin{array}{cccccccc}
\text{Coord. of Stations and Center of Mass} & \text{State Vector, Pass 1} & \text{Error Coeff., Pass 1} & \text{State Vector, Pass 2} & \text{Error Coeff., Pass 2} & & \text{State Vector, Pass n} & \text{Error Coeff., Pass n}
\end{array}\\
\begin{bmatrix}
\sum_{i=1}^{n} \hat{U}_i & \dot{U}_1 & \bar{U}_1 & \dot{U}_2 & \bar{U}_2 & \ldots & \dot{U}_n & \bar{U}_n \\
\dot{U}_1^T & \dot{N}_1 & \bar{N}_1 & 0 & 0 & \ldots & 0 & 0 \\
\bar{U}_1^T & \bar{N}_1^T & \ddot{N}_1 & 0 & 0 & \ldots & 0 & 0 \\
\dot{U}_2^T & 0 & 0 & \dot{N}_2 & \bar{N}_2 & \ldots & 0 & 0 \\
\bar{U}_2^T & 0 & 0 & \bar{N}_2^T & \ddot{N}_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\dot{U}_n^T & 0 & 0 & 0 & 0 & \ldots & \dot{N}_n & \bar{N}_n \\
\bar{U}_n^T & 0 & 0 & 0 & 0 & \ldots & \bar{N}_n^T & \ddot{N}_n
\end{bmatrix}
\end{array}
\begin{bmatrix}
\hat{\delta} \\ \dot{\delta}_1 \\ \ddot{\delta}_1 \\ \dot{\delta}_2 \\ \ddot{\delta}_2 \\ \vdots \\ \dot{\delta}_n \\ \ddot{\delta}_n
\end{bmatrix}
=
\begin{bmatrix}
\sum_{i=1}^{n} \hat{c}_i \\ \dot{c}_1 \\ \ddot{c}_1 \\ \dot{c}_2 \\ \ddot{c}_2 \\ \vdots \\ \dot{c}_n \\ \ddot{c}_n
\end{bmatrix}.
$$

Having formed the system of normal equations for the adjustment, we must now address the problem of solving the system, expecially in view of the consideration that it can grow to huge dimensions. We do this in Section 4 which is devoted to the theory of partitioned regression. We need only point out here that the system of normal equations (52) is precisely of the same form as the second order partitioned system indicated in equation (66) of Section 4. Accordingly, by applying the algorithms of second order partitioned regression as developed in Section 4 to the present problem, one can execute a practical solution of the normal equations no matter how many arcs are processed or how many error parameters are introduced by each arc.

2.8 Analysis of Residuals

After the normal equations have been solved by the algorithm for second order partitioned regression, the optical residuals can be computed from:

$$v_j = \begin{Bmatrix} v_x \\ v_y \\ v_t \end{Bmatrix}_j = \Lambda_j A_j^T (A_j \Lambda_j A_j^T)^{-1} (\epsilon_j - B_j \delta). \tag{53}$$

These residuals are employed in SAGA, along with residuals corresponding to other observed quantities, to compute:

$$s = \left[\frac{\text{(quadratic form of all residuals)}}{\text{(degrees of freedom)}}\right]^{\frac{1}{2}} .$$

The observational equations are then relinearized about the new approximations to the parameters and the solution is iterated, leading to fresh residuals and a new value for s. This process is repeated until successive values of s differ by less than a preset criterion or the maximum allowable number of iterations have been executed. Computational details are given in Section 6.

Upon convergence, the final observational residuals from designated channels can, on option, be subjected to an autoregressive analysis. When this option is exercised, the entire solution is repeated with serial correlation being duly considered in accordance with the process of autoregressive feedback developed in Section 5.

2.9 Master Survey File

Before the adjustment of satellite observations can be undertaken, it is necessary to set up a Master Survey File for the entire tracking net ultimately to be adjusted. This file contains the initial coordinates of all stations and the covariance matrix of the coordinates.

When initial coordinates are expressed as geographical coordinates, a preliminary program (SET-UP) transforms these coordinates into Geocentric Cartesian coordinates and computes the associated covariance matrix. In order to permit the investigator considerable flexibility in imposing interstation constraints, SET-UP also permits the directional components and the length of the vector joining arbitrary pairs of stations to be constrained to any desired degree. In addition, it permits the introduction of any number of linear constraints between arbitrary pairs of stations. Thus if $(X_p^{00}, Y_p^{00}, Z_p^{00})$, $(X_q^{00}, Y_q^{00}, Z_q^{00})$ denote the initial coordinates of stations p and q, the program admits constraints of the form:

$$\alpha_1 X_p^{00} + \alpha_2 Y_p^{00} + \alpha_3 Z_p^{00} + \beta_1 X_q^{00} + \beta_2 Y_q^{00} + \beta_3 Z_q^{00} = U_{pq} \tag{54}$$

in which U_{pq} is considered to be an observation of prespecified variance σ^2_{pq} and the α's and β's are prespecified coefficients. Such constraints serve a variety of functions, including (a) holding selected stations fixed relative to a designated station (datum constraints), (b) defining directions of coordinate axes (for adjustments limited to ranging observations), (c) imposing special relations between stations.

All specified interstation constraints are properly exercised by SET-UP in developing the a priori covariance matrix $\hat{\Lambda}$ of the geocentric coordinates of the tracking net. Details of this process are given in Part II. The location of the center of mass with respect to the adopted geometric origin is treated as if it were the last tracking station of the net. By virtue of this artifice, the covariance matrix $\hat{\Lambda}$ serves to accommodate the a priori covariance matrix of the coordinates of the center of mass.

The Master Survey File is set up but once for a given tracking net. SAGA calls upon this file whenever it requires either the a priori coordinates of a given station or the a priori weight matrix $(\hat{W} = \Lambda^{-1})$ of the entire vector of station coordinates.

3.0 ELECTRONIC RANGING OBSERVATIONS

3.1 The Geoceiver

In this section we shall develop observational equations and error models appropriate to various electronic ranging systems: lasers, Secor, GRARR, Radars, and Geoceiver. We shall place particular emphasis on the Geoceiver system both because of its potential future importance to satellite geodesy and because its error model turns out to be sufficiently broad to encompass those of all other ranging systems.

Characteristics of the Geoceiver are discussed in Stansell, et. al. (1965). Briefly, the Geoceiver is a compact, self-contained, relatively inexpensive (under $100K), manpack, doppler tracking unit designed to track Transit satellites as well as other satellites radiating on either of the frequency pairs: 162/324M Hz, 150/400M Hz. Reception of dual frequencies provides the means for correction of ionospheric refraction. Like the Tranet system, the Geoceiver exploits one way doppler. However, Tranet operates on cycle counts in a destructive mode; that is, at preset intervals it measures the time required to acquire a preset number of doppler cycles, whereupon, it ceases counting until the start of the next interval. Thus, continuity of cycle count is lost and it becomes necessary to treat Tranet observations as being essentially a measure of doppler frequency (or, equivalently, of range rate). By contrast, Geoceiver operates on cycle counts in a nondestructive mode; that is, continuity of cycle count is preserved (as long as phase lock is maintained). It is this fact that permits Geoceiver to be viewed as being inherently a ranging system, for if the transmitted frequency from the satellite and the reference frequency generated by Geoceiver were perfectly matched and both were perfectly stable, the scaled cumulative cycle count of beat frequency would, except for an unknown additive constant of integration, represent a direct measure of slant range. In practice, this measure is contaminated by the unknown offset between transmitted frequency and local reference frequency as well as by any drifts of the two frequencies. Such factors can, however, be taken into account by error modelling.

3.2 Geoceiver Observational Equations and Error Model

To arrive at the Geoceiver observational equation we find it convenient first to define the quantities:

r = distance of satellite from station at time t (station clock)

f_o = frequency transmitted from satellite

f = frequency received at station at time t

v = total velocity of satellite at time t

θ' = angle between velocity vector $\bar{v}$ and vector directed from observer to satellite at time t ($\cos \theta' = \bar{r} \cdot \bar{v}$)

$\dot{r}$ = rate of change of range at time $t = v \cos \theta'$.

The relativistic equation describing the Doppler effect on frequency is then of the form (JOOS, THEORETICAL PHYSICS):

$$(1) \quad \frac{f}{f_o} = \left(1 - \frac{v}{c} \cos \theta'\right) \Big/ \sqrt{1 - \left(\frac{v}{c}\right)^2} \; .$$

Setting $v \cos \theta' = \dot{r}$ and solving for $\dot{r}$, we get:

$$(2) \quad \dot{r} = c \left[1 - \frac{f}{f_o} \sqrt{1 - \left(\frac{v}{c}\right)^2} \right]$$

$$= c \, \frac{f_o - f}{f_o} + \frac{1}{2} \frac{v^2}{c} + \text{higher order terms.}$$

Let f_o' denote the reference frequency generated by the Geoceiver. Then (2) can be written:

$$(3) \quad \dot{r} = \lambda \left[\Delta f + (f_o - f_o') \right] + \frac{1}{2} \frac{v^2}{c}$$

where

$$(4) \quad \begin{aligned} \lambda &= c/f_o \\ \Delta f &= f_o' - f \, . \end{aligned}$$

The quantity λ represents the wavelength of the transmitted frequency and the quantity represents the beat frequency generated by the Geoceiver. If we now assume that f_0 and f_0' are constant over the tracking interval, we can integrate both sides of (3) between an arbitrary time $\tau = 0$ and a later time τ to obtain:

$$(5) \qquad r - r_0 = \lambda N + \lambda(f_0' - f_0)\tau + \Delta r_s$$

in which

r = range at time τ,

r_0 = range at time $\tau = 0$,

$N = \int_0^{\tau} \Delta f \, dt$,

$\Delta r_s = \frac{1}{2c} \int_0^{\tau} v^2 \, dt$ = special relativistic correction.

The integral defining N represents the number of cycles of beat frequency accumulated from the initiation of counting ($\tau = 0$) until time τ.

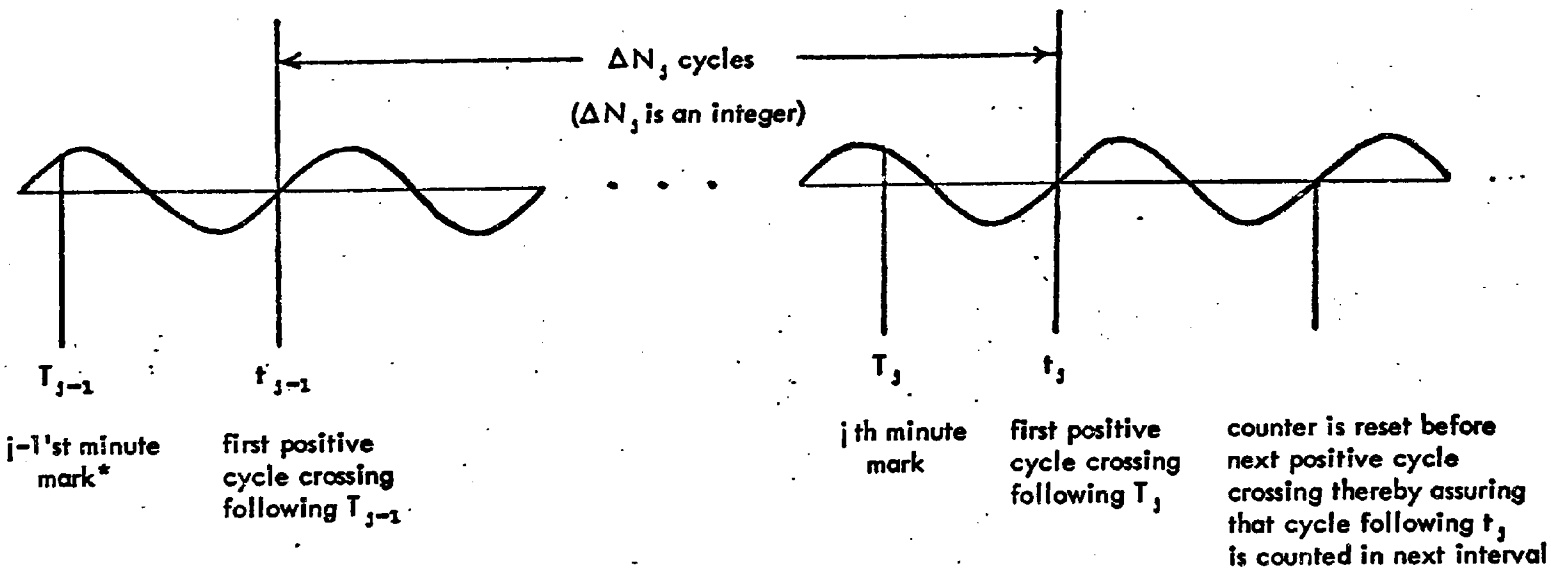
In the case of Geoceiver, cycle counts are cumulated over intervals of nominally one minute. Specifically, Geoceiver generates the cycle counts:

$$(6) \qquad \Delta N_j = \int_{\tau_{j-1}}^{\tau_j} \Delta f \, dt$$

in which τ_{j-1} and τ_j represent the times of the first positive cycle crossings following successive one minute marks T_{j-1} and T_j, as shown in the accompanying figure. Following readout, the counter is reset to zero before the next positive cycle crossing is counted in the next interval and continuity of cycle count is maintained. The cycle count N appearing in (5) can be related to the counts generated in (6) by the relation:

$$(7) \qquad N_j = \int_0^{\tau_j} \Delta f \, dt = \int_0^{\tau_1} \Delta f \, dt + \int_{\tau_1}^{\tau_2} \Delta f \, dt + \ldots + \int_{\tau_{j-1}}^{\tau_j} \Delta f \, dt.$$

$$= \Delta N_1 + \Delta N_2 + \ldots + \Delta N_j .$$



*NOTE: When Transit satellites are tracked, one minute marks for triggering readout are generated by interpolation within the standard two minute synchronization words impressed on the received signal; otherwise, one minute marks are generated internally.

FIGURE 3. Illustrating method used by Geoceiver to obtain continuous count of cycles of beat frequency.

One might expect from (7) that errors in successive N_j are highly correlated by virtue of being formed of common ΔN_i. This is not the case, however, when Geoceiver is functioning properly (i.e., no cycles are dropped from or added to each count) for the ΔN_j, being integers, may be viewed as being free of error. The quantity properly to be regarded as subject to error is τ_j, the time of the end of each counting interval. Errors in successive times τ_j and τ_{j-1} are not likely to be strongly correlated.

In practice, the frequencies f_0 and f_0' are not perfectly known, nor are they perfectly stable. To account for biases and linear drifts in these frequencies we may write:

$$
(8) \qquad
\begin{aligned}
f_0 &= f_{00} + \delta f_0 + \dot{f}_0 \tau \\
f_0' &= f_{00}' + \delta f_0' + \dot{f}_0' \tau
\end{aligned}
$$

in which

f_{00} = adopted value of transmitted frequency

f_{00}' = adopted value of local reference frequency

δf_0 = bias in adopted value of f_0 at $\tau = 0$

$\delta f_0'$ = bias in adopted value of f_0' at $\tau = 0$

$\dot{f}_0, \dot{f}_0'$ = drift rates of f_0 and f_0', respectively.

For Geoceiver, the offset $\Delta f_{00} = f_{00}' - f_{00}$ between the adopted frequencies can range between 16 Kc and 32 Kc, depending on the type of satellite. Because of this and the fact that doppler ranges between ±10 Kc, the beat frequency Δf will never cross through zero. This obviates the need for distinguishing between positive and negative cycles. If we substitute (8) into (3) and expand the resulting expression for $1/f_0$ into the series

$$
(9) \qquad \frac{1}{f_0} = \frac{1}{f_{00}} \left(1 - \frac{\delta f_0}{f_{00}} - \frac{\dot{f}_0}{f_{00}} \tau + \ldots \right)
$$

we shall obtain the result

$$(10)\quad \dot{r} = \lambda_0 \left(1 - \frac{\delta f_0}{f_\infty}\right)(\Delta f - \Delta f_\infty) + \lambda_0 (\delta f_0 - \delta f_0') + \lambda_0 (\dot{f}_0 - \dot{f}_0')\tau + \frac{1}{2}\,\frac{v^2}{c} + \text{higher order terms,}$$

where

$$(11)\quad \lambda_0 = \frac{c}{f_{00}} = \text{wavelength corresponding to adopted frequency of transmission.}$$

Integrating this expression, as before, we get:

$$(12)\quad r - r_0 = \lambda_0\left(1 - \frac{\delta f_0}{f_\infty}\right)(N - \Delta f_\infty\, \tau) + \lambda_0 (\delta f_0 - \delta f_0')\,\tau + \frac{1}{2}\lambda_0 (\dot{f}_0 - \dot{f}_0')\,\tau^2$$

$$+ \Delta r_s + \text{higher order terms.}$$

This can be put into the form:

$$(13)\quad r = \lambda_0 (N - \Delta f_\infty\, \tau) + a_0 + a_1\tau + a_2\tau^2 + a_3 r + \Delta r_s + \text{higher order terms}$$

where:

(14)

$$a_0 = \left(1 + \frac{\delta f_0}{f_\infty}\right) r_0$$

$$a_1 = \lambda_0 (\delta f_0 - \delta f_0')$$

$$a_2 = \frac{1}{2}\lambda_0 (\dot{f}_0 - \dot{f}_0')$$

$$a_3 = -(\delta f_0 / f_\infty).$$

*If satellite clock triggers readout, replace δf_0 by $\left(1 + \frac{\dot{r}}{c}\right)\delta f_0$

If Geoceiver clock triggers readout, replace $\delta f_0'$ by $\left(1 + \frac{\dot{r}}{c}\right)\delta f_0'$

Substituting:

$$(15)\quad r = \left[(X - X^c)^2 + (Y - Y^c)^2 + (Z - Z^c)^2\right]^{\frac{1}{2}}$$

$$(16)\quad r^0 = \lambda_0 (N - \Delta f_\infty\, \tau)$$

into (13) we obtain the basic equation

* See equation (25) for explanation.

(17) $r^0 + a_0 + a_1\tau + a_2\tau^2 + a_3 r + \delta\dot{r}_s = [(X-X^c)^2 + (Y-Y^c)^2 + (Z-Z^c)^2]^{\frac{1}{2}}$.

The development thus far has failed to take into account the refractive effects of the ionosphere and troposphere. Nor has it taken into account the effects of (a) propagation delay, (b) interstation timing bias and (c) general relativistic effects. Thus to render (17) more nearly correct we should add to r^0 the composite correction:

(18) $\Delta r = \Delta r_I + \Delta r_T + \Delta r_p + \Delta r_\tau + \Delta r_\epsilon$

where

Δr_I = correction for ionospheric refraction

Δr_T = correction for tropospheric refraction

Δr_p = correction for propagation delay

Δr_τ = correction for interstation timing offset (or bias)

Δr_G = correction for effect of gravitational potential or frequency (general relativistic effect).

The two frequency method is used by Geoceiver to determine ionospheric refraction. If $\Delta\Delta N_j$ denotes the refraction cycle count (i.e., the cycle count of the beat frequency between the two received frequencies), the desired correction is given by:

(19) $\Delta r_I = -K\,\lambda_0(\Delta\Delta N_1 + \Delta\Delta N_2 + \ldots + \Delta\Delta N_j)$

where $K = 1/9$ for the frequency pair 162/324 Mc and $K = 1/9\frac{1}{6}$ for the frequency pair 150, 400Mc. Alternatively, the correction could be effected by replacing each ΔN_j in (7) by $\Delta N_j - K\,\Delta\Delta N_j$.

The correction for tropospheric refraction can be computed by the following formula derived by J. Willmann (private communication):

(20) $\Delta r_T = -2\alpha\, f(E)$

where

$$\alpha = (n_0 - 1)\, H_0$$

(21)

$$f(E) = 1 \Big/ \left[\sin E + \left(\sin^2 E + \frac{4H_0}{r_0}\right)^{\frac{1}{2}}\right]$$

in which

n_0 = index of refraction at station

r_0 = radius of earth (meters)

H_0 = scale height of troposphere (meters).

The value of n_0 can be computed from meteorological measurements at the station by means of the formula:

$$n_0 = 1 + 10^{-6} \left\{103.49 \left(\frac{P_0 - e_0}{T_0}\right) + \frac{86.26}{T_0} \left(1 + \frac{57.48}{T_0}\right) e_0\right\} \tag{22}$$

where

P_0 = atmospheric pressure (mm/Hg)

e_0 = water vapor pressure (mm/Hg)

T_0 = temperature (deg Kelvin).

The scale height H_0 can be approximated by:

$$H_0 = 29.2\,(T_0 - 30). \tag{23}$$

The correction for propagation delay serves to correct the range of the satellite to the position occupied at time τ_j as measured by the satellite clock (this is equivalent to the range at time $\tau_j + \frac{r}{c}$ as measured by the station clock):

$$\Delta r_p = \dot{r}\,\frac{r}{c}\,. \tag{24}$$

When a Geoceiver station is considered as one of several stations participating in the tracking of a pass, it becomes necessary to allow for the possibility of a significant bias in the correlation between the clocks at the various stations. If $\delta\tau_0$ denotes the offset at epoch of the clock at a particular Geoceiver station from the master clock, the offset at time τ is:

$$\delta\tau = \delta\tau_0 + \frac{\delta f_0}{f_{\infty}}\tau = \delta\tau_0 + \lambda_0 \frac{\delta f_0}{c}\tau$$

(for readout triggered by timing signals from the satellite clock), and the correction to be applied to r^0 is:

$$(25) \qquad \Delta r_{\tau} = \dot{r}\,\delta\tau = \dot{r}\,\delta\tau_0 + \lambda_0\,\delta f_0\,\frac{\dot{r}}{c}\,\tau\,.$$

When the offset is unknown, $\delta\tau_0$ can be carried as an unknown parameter, subject to appropriate a priori constraints. The contribution of δf_0 can be absorbed into the error coefficient a_1. For readout triggered by the local clock, δf_0 in (25) should be replaced by $\delta f_0'$.

The final correction is attributable to the effects on frequency of the difference in gravitational potential between the satellite and the receiving station. The correction is given by:

$$(26) \qquad \Delta r_g = \frac{\mu}{c\,r_0}\int_0^{\tau}\frac{(h/r_0)\,d\tau}{(1+h/r_0)}$$

where

μ = gravitational constant $\approx 4\times10^{14}\ m^3/sec^2$

h = altitude of satellite

r_0 = distance of station from center of earth (same units as h).

It is to be noted that for nearly circular orbits, h is nearly constant over a pass. Consequently, for such orbits Δr_g becomes a linear function of τ, and the term in a_1 in (17) can absorb the effects of the general relativistic correction. A similar remark applies to the special relativistic correction Δr_s inasmuch as velocity v is nearly constant for nearly circular orbits.

For greater flexibility we shall augment the Geoceiver error model by terms to account for residual interstation timing bias and residual tropospheric refraction (we assume that the corrections indicated above have been applied but are not wholly adequate). Thus the full error model becomes:

$$(27) \qquad \delta r = a_0 + a_1\tau + a_2\tau^2 + a_3 r + a_4\dot{r} + a_5 f(E)$$

where the terms in a_4 and a_5 account for interstation timing bias and residual tropospheric refraction, respectively. Regarding residual tropospheric refraction we would remark that ray tracing exercises through actual atmospheric profiles indicate that if the best possible value of α were employed in (20), the proportional error $\delta R_T/R_T$ in the tropospheric refraction correction could be held to under one percent for $E \geq 10°$ (or to about 0.2 m at $E=10°$). However, when α is computed from meteorological measurements made at the station, $\delta R_T/R_T$ is likely to range from three to five percent for $E \geq 10°$ (and thus become as great as 1.0m at $E=10°$). Hence, potential improvement by a factor of from three to five is possible from error modeling.

3.3 Linearized Observational Equations

We have already remarked that the quantity subject to error in Geoceiver observations consists actually of the time associated with the cycle count (here, we ignore momentarily the contribution of errors in corrections for ionospheric refraction). Let us suppose that for a given value of r^0 in (17) the measured time is τ and the correct time is $\tau - \epsilon_\tau$ (thus ϵ_τ denotes the error in τ). Then we can write

(28) $$r^0(\tau - \epsilon_\tau) = r^0(\tau) - \dot{r}(\tau)\,\epsilon_\tau + \text{higher order terms.}$$

Now, ϵ_τ is attributable in part to errors in phase measurement and in part to errors in quantization. Thus, we may write:

(29) $$\epsilon_\tau = \epsilon'_\tau + \epsilon''_\tau$$

where

ϵ'_τ = contribution of phase measuring error

ϵ''_τ = contribution of quantization error.

If ϵ_φ denotes the phase measuring error (expressed as a fraction of a cycle) giving rise to ϵ'_τ, we may write

(30) $$\epsilon'_\tau = \frac{\epsilon_\varphi}{\Delta f_\tau}$$

where Δf_τ denotes the beat frequency at time τ. But from (10) $\Delta f_\tau \approx \frac{1}{\lambda_0} \dot{r}(\tau)$, so that

(31) $\epsilon'_\tau = \lambda_0 \epsilon_\varphi / \dot{r}(\tau)$.

Substituting this into (29) and substituting the resulting expression into (28), we get:

(32) $r^0(\tau - \epsilon_\tau) = r^0(\tau) - \lambda_0 \epsilon_\varphi - \dot{r}(\tau) \epsilon''_\tau$.

Letting v_r and v_τ denote residuals corresponding to the errors $\lambda_0 \epsilon_\varphi$ and ϵ''_τ, we may write the Geoceiver observational equations in the expanded form:

(33)
$$r^0 + v_r + \dot{r} v_\tau + \Delta r + a_0 + a_1\tau + a_2\tau^2 + a_3 r + a_4 \dot{r} + a_5 f(E) =$$
$$[(X-X^c)^2 + (Y-Y^c)^2 + (Z-Z^c)^2]^{\frac{1}{2}}.$$

If we now replace X, Y, Z on the rig[illegible] hand side of (33) with the values computed from the orbital integrator (eqs (24) and (21) of Sec. 2), we shall obtain an equation of the functional form:

(34) $r = f(X^c, Y^c, Z^c, X_{00}, Y_{00}, Z_{[illegible]} \ldots, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0, t)$

where r denotes the left side of (33).

From this point on we proceed precisely as we did in Section 3 with optical observations. We express the unknowns in (34) in terms of the same initial approximations and corrections as were used in the optical reduction (eq. (29), Sec. 3), and thus expand the resulting expression in Taylor's series to get:

(35)
$$r = r^{00} + \frac{\partial r}{\partial(X^c, Y^c, Z^c, X_{00}, Y_{00}, Z_{00}, x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)} (\delta X^c, \delta Y^c, \delta Z^c, \ldots, \delta\dot{y}_0, \delta\dot{z}_0)^T$$

in which

(36) $r^{00} = f((X^c)^{00}, (Y^c)^{00}, \ldots, \dot{z}_0^{00}, \tau)$.

If we let $X^{00}, Y^{00}, Z^{00}, \dot{X}^{00}, \dot{Y}^{00}, \dot{Z}^{00}$ denote the earth fixed coordinates of the satellite as determined from the integration based on the approximate state vector, the partial derivatives of r with respect X^c, Y^c, Z^c can be expressed as:

$$(37) \quad \underset{(1,3)}{B^{(1)}} = \frac{\partial r}{\partial(X^c, Y^c, Z^c)} = -[\lambda \; \mu \; \nu]$$

in which

$$\lambda = (X^{oo} - (X^c)^{oo})/r^{oo}$$

$$(38) \quad \mu = (Y^{oo} - (Y^c)^{oo})/r^{oo}$$

$$\nu = (Z^{oo} - (Z^c)^{oo})/r^{oo}.$$

The partial derivatives of r with respect to center of mass and orbital state vector are then given by:

$$(39) \quad \underset{(1,9)}{B^{(3)}} = \frac{\partial r}{\partial(X_{oo}, Y_{oo}, \ldots, \dot{z}_0)} = - \underset{(1,3)}{B^{(1)}} \underset{(3,9)}{\Phi}$$

where Φ is given by (26) of Sec. 3. The approximate value of $\dot{r}$ can be computed from:

$$(40) \quad \dot{r}^{oo} = - \underset{(1,3)}{B^{(1)}} \begin{bmatrix} \dot{X}^{oo} \\ \dot{Y}^{oo} \\ \dot{Z}^{oo} \end{bmatrix}.$$

If we partition $B^{(3)}$ as:

$$(41) \quad \underset{(1,9)}{B^{(3)}} = \left[\frac{\partial r}{\partial(X_{oo}, Y_{oo}, Z_{oo})} \;\middle|\; \frac{\partial r}{\partial(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)} \right] = \left[\underset{(1,3)}{B^{(3a)}} \;\middle|\; \underset{(1,6)}{B^{(3b)}} \right]$$

The linearized observation equation can be expressed as:

$$(42) \quad \underset{(1,2)}{A} \; \underset{(2,1)}{v} - \underset{(1,3)}{B^{(1)}} \; \underset{(3,1)}{\delta^{(1)}} \; \underset{(1,3)}{B^{(2a)}} \; \underset{(3,1)}{\delta^{(2a)}} - \underset{(1,6)}{B^{(2b)}} \; \underset{(6,1)}{\delta^{(2b)}} - \underset{(1,5)}{B^{(3)}} \; \underset{(5,1)}{\delta^{(3)}} - \underset{(1,1)}{B^{(4)}} \; \underset{(1,1)}{\delta^{(4)}} = \underset{(1,1)}{\epsilon}$$

in which

$$A = [1 \;\; \dot{r}^{\infty}], \quad v = \begin{bmatrix} v_r \\ v_\tau \end{bmatrix}, \quad \delta^{(1)} = \begin{bmatrix} \delta X^c \\ \delta Y^c \\ \delta Z^c \end{bmatrix}, \quad \delta^{(2a)} = \begin{bmatrix} \delta X_{\infty} \\ \delta Y_{\infty} \\ \delta Z_{\infty} \end{bmatrix}, \quad \delta^{(2b)} = \begin{bmatrix} \delta x_o \\ \delta y_o \\ \vdots \\ \delta y_o \end{bmatrix}$$

(43)

$$B^{(3)} = [\tau \;\; \tau^2 \;\; r \;\; \dot{r} \;\; f(E)], \quad \delta^{(3)} = (a_1 \;\; a_2 \;\; a_3 \;\; a_4$$

$$B^{(4)} = (1), \quad \delta^{(4)} = (a_0).$$

Before proceeding, we shall address one possible problem with Geoceiver tracking that has not so far been considered. This is concerned with what to do in the case of one or more temporary tracking dropouts during a pass. When phase lock is lost, the doppler counter is immediately set to zero and remains set to zero until the first one minute mark after phase lock is restored. Thus a zero cycle count for a given counting interval is a positive indication of a dropout during that interval. When counting is resumed following a dropout, a new constant of integration r_0, or equivalently a new error coefficient a_0, must be established. On the other hand, it is clear from their physical interpretation that the remaining error coefficients $(a_1, a_2, a_3, a_4, a_5)$ will not be altered by a dropout. Hence, only a_0 requires reinitialization following a dropout. For each pass provisions are made in SAGA to accommodate up to a maximum of three dropouts in tracking from each participating station. This is accomplished by writing the observational equation generated by the jth point as:

(44) $\quad A_j \, v_j + B_j \, \delta = \epsilon_j$

in which

$$
\begin{array}{lcccccccccc}
 & & \text{Station Coord.} & \text{Center of Mass} & \text{Orbital State Vector} & \text{Error Coeff.} & \text{Zero Set, Interval 1} & \text{Zero Set, Interval 2} & \text{Zero Set, Interval 3} & \text{Zero Set, Interval 4} & \\
(45) \quad \underset{(1,\,12+\ell)}{B_j} & = -[& \underset{(1,3)}{B_{pj}^{(1)}} & \underset{(1,3)}{B_{pj}^{(2a)}} & \underset{(1,6)}{B_{pj}^{(2b)}} & \underset{(1,8)}{B_{pj}^{(3)}} & \underset{(1,1)}{\xi_{1p} B_{pj}^{(4)}} & \underset{(1,1)}{\xi_{2p} B_{pj}^{(4)}} & \underset{(1,1)}{\xi_{3p} B_{pj}^{(4)}} & \underset{(1,1)}{\xi_{4p} B_{pj}^{(4)}} &]
\end{array}
$$

$$
\begin{array}{lcccccccccc}
(46) \quad \underset{(1,\,12+\ell)}{\delta^T} & = (& \underset{(1,3)}{\delta^{(1)^T}} & \underset{(1,3)}{\delta^{(2a)^T}} & \underset{(1,6)}{\delta^{(2b)^T}} & \underset{(1,4)}{\delta^{(3)^T}} & \underset{(1,1)}{\delta_1^{(4)^T}} & \underset{(1,1)}{\delta_2^{(4)^T}} & \underset{(1,1)}{\delta_3^{(4)^T}} & \underset{(1,1)}{\delta_4^{(4)^T}} &)^T
\end{array}
$$

in which the subscript p connotes the pth tracking interval (max p=4) and

$$
(47) \qquad
\begin{aligned}
\xi_{ip} &= 1 \quad \text{if} \quad i=p \\
\xi_{ip} &= 0 \quad \text{if} \quad i \neq p .
\end{aligned}
$$

The dimension ℓ indicates the number of error coefficients exercised by the station on the pass (this can range from a minimum of six to a maximum of nine depending on the number of tracking dropouts).

3.4 Normal Equations

The development of the normal equations generated by Geoceiver observations follows precisely the process outlined in Section 2.7 for optical observations. With suitable (and perfectly obvious) reinterpretation, equations (44) through (49) hold also for Geoceiver observations. In particular, the partial set of normal equations generated by a given station for a given pass (eq. (49) of Sec. 2.7) applies equally to Geoceiver observations. So also does the merged system of equations from all stations for a given pass (eq. (50), Sec. 2.7), with one proviso, namely, that each Geoceiver station be assigned a distinct station number even when it is colocated with an optical station.

This step automatically accounts for the fact that the vector $\ddot{\delta}_{ijk}$ of error coefficients for a Geoceiver is different from the vector $\ddot{\delta}_{ijk}$ of error coefficients for a camera. Should a Geoceiver be colocated with a camera, appropriate interstation constraints can be exercised to insure that both stations receive a common adjustment. With this understanding then, we can regard the general normal equations (52) as applying equally well to (a) a network of optical tracking, (b) a network of Geoceivers (or other ranging systems), (c) a combined network of optical trackers and Geoceivers.

3.5 A Priori Constraints

Of the six coefficients in the Geoceiver error model, the first two a_0 and a_1 have large effects and are unlikely to be subject to worthwhile a priori constraints. The remaining four, on the other hand, have relatively small effects and are subject to rather tight a priori constraints. When the Geoceiver is functioning properly and when the frequency bias δf_0 of the satellite oscillator is reasonably well monitored, the a priori constraints indicated in the table below can serve as rough guidelines.

Coefficient	Physical Significance	Typical A Priori Constraint (One Sigma)
a_0	$\left(1 + \frac{\delta f_0}{f_{00}}\right) r_0$	10^6 to 10^7 m
a_1	$\lambda_0 (\delta f_0 - \delta f_0')$	$10\lambda_0$ to $10^3 \lambda_0$ m/sec
a_2	$\frac{1}{2} \lambda_0 (\dot{f}_0 - \dot{f}_0')$	$\frac{1}{2} \lambda_0 f_0 \times 10^{-11}$ m/sec^2
a_3	$-\delta f_0 / f_{00}$	10^{-6} to 10^{-7}
a_4	$\delta\tau$ (timing bias)	$50 \mu s$
a_5	$\delta\alpha$ (refraction)	0.2m

In our view, the major shortcoming of Geoceiver is its low sampling rate of only one readout per minute. This means that many passes will yield under ten observations and even the longest passes are unlikely to yield more than twenty observations. The justification of the low sampling rate probably stems from the use (in one mode of operation) of Transit timing signals to trigger readout of cycle count. Also, a probable factor is the approach to data reduction envisioned by the designers of Geoceiver. As outlined in Stansell, et. al. (1965), this approach involves using an observational equation of the form:

$$(48) \qquad r_j - r_{j-1} = \lambda_0 \Delta N_j + (f_0 - f_0')(\tau_j - \tau_{j-1}) + \text{neglected higher order terms,}$$

which is relatively weak, geometrically, for small time intervals. This formulation has the advantage of completely eliminating the error coefficient for zero set a_0 and of generally suppressing to insignificance the effects of a_2, a_3, a_4, a_5 (over intervals as short as one minute). On the other hand, it fails to exploit one of the strongest characteristics of the Geoceiver, namely, the nondestructive readout of cycle counts. It is clear that (48) is equally valid for destructive or nondestructive readout of cycle counts, whereas our formulation (eq. (13)) is designed specifically to exploit the nondestructive character of the readout. If readout is indeed nondestructive, our approach should result in considerably more effective utilization of Geoceiver observations. On the other hand, our approach could benefit considerably from an increased sampling rate. An internally triggered readout every ten or twenty seconds would probably be close to ideal. Such serial correlation as might thereby be introduced could be properly taken into account by the process of auto-regressive feedback developed in Section 5.

With the readout rate as low as it is, one cannot generally expect to obtain from the adjustment any significant degree of improvement (over a priori values) in the adjusted error coefficients a_3, a_4, a_5. Here, what one mainly accomplishes is the realization of a more valid error propagation by considering the effects of these sources of error. The a priori value of a_2 is subject to a degree of improvement ranging from slight for short passes (under 5 minutes) to pronounced for moderately long passes (over 15 minutes). On long passes observed by several stations (five or more), one can expect to determine values of a_1 and a_2 to accuracies (sigma) of better than five meters and five millimeters per second, respectively.

3.6 Reduction of Other Ranging Systems

Although emphasis in this section has been on the Geoceiver, our treatment of the Geoceiver as a ranging system makes it possible to employ SAGA for the reduction of ranging systems in general. The primary requirement for processing other ranging observations through SAGA is that they be presented to the program in the Geos data format. With active ranging systems, the coefficient a_0 would ordinarily be subject to very tight a priori constraints (e.g., a few meters for lasers), and the coefficients a_1, a_2 would be inapplicable and hence would be suppressed. An option is provided within SAGA for the application of corrections for tropospheric refraction in the event that such corrections are lacking.

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4.0 THEORY AND EXECUTION PARTITIONED REGRESSION

4.1 First Order Partitioned Regression

SAGA is designed to process an arbitrarily large number of passes, each observed by a subset of as many as fifteen stations each of which can conceivably (with re-initialization of error coefficients) introduce as many as seventeen error coefficients peculiar to the pass. Accordingly, SAGA generates a patterned system of normal equations that grows in dimensions both with the number of passes processed and with the number of error parameters exercised on each pass. In practical applications, the normal equations can grow to embrace thousands of unknowns and thus be unamenable to solution by conventional reductions that fail to exploit the patterned characteristics of the system. In this section we shall develop practical agorithms for the solution of normal equations related to those generated by SAGA.

The normal equations for a general regression analysis may be written as

(1) $$N\delta = c$$

in which δ denotes the parametric vector to be estimated, N is the coefficient matrix, and c is the constant column. Let δ be arbitrarily be partitioned into two vectors $\dot{\delta}$, $\ddot{\delta}$ so that

(2) $$\delta = \begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} .$$

Then the original normal equations partitioned to be conformable with the partitioning of δ can be written as

(3) $$\begin{bmatrix} \dot{N} & \bar{N} \\ \bar{N}^T & \ddot{N} \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{c} \\ \ddot{c} \end{bmatrix} .$$

To this point the normal equations are of perfectly general form. We now abandon full generality by assuming that the parametric vector $\dot{\delta}$ is common to most, if not all, of the original observational equations, whereas $\ddot{\delta}$ is composed of (possibly) a large number of subvectors $\ddot{\delta}_1$, $\ddot{\delta}_2, \ldots, \ddot{\delta}_n$, each of which appears only in a subgroup of observational equations to the exclusion of other subvectors of $\ddot{\delta}$. Under these circumstances, the normal equations (3) assume the specialized form

$$
(4)\qquad
\left[\begin{array}{c|cccc}
\dot{N} & \bar{N}_1 & \bar{N}_2 & \dots & \bar{N}_n \\
\hline
\bar{N}_1^T & \ddot{N}_1 & 0 & \dots & 0 \\
\bar{N}_2^T & 0 & \ddot{N}_2 & \dots & 0 \\
\vdots & \vdots & \vdots & \dots & 0 \\
\bar{N}_n^T & 0 & 0 & \dots & \ddot{N}_n
\end{array}\right]
\left[\begin{array}{c}
\dot{\delta} \\
\hline
\ddot{\delta}_1 \\
\ddot{\delta}_2 \\
\vdots \\
\ddot{\delta}_n
\end{array}\right]
=
\left[\begin{array}{c}
\dot{c} \\
\hline
\ddot{c}_1 \\
\ddot{c}_2 \\
\vdots \\
\ddot{c}_n
\end{array}\right] .
$$

Here, the $\ddot{N}$ portion of the matrix assumes a block diagonal form by virtue of the assumption that for all $i \neq j$, $\ddot{\delta}_i$ and $\ddot{\delta}_j$ do not appear in common observational equations.

To derive this result we may proceed as follows. The linearized observational equations may be ordered into groups corresponding to the various parametric subvectors $\ddot{\delta}_i$. Thus we write

$$
(5)\qquad
\begin{aligned}
v_1 + \dot{B}_1 \dot{\delta} + \ddot{B}_1 \ddot{\delta}_1 &= \epsilon_1 \\
v_2 + \dot{B}_2 \dot{\delta} + \ddot{B}_2 \ddot{\delta}_2 &= \epsilon_2 \\
&\vdots \\
v_n + \dot{B}_n \dot{\delta} + \ddot{B}_n \ddot{\delta}_n &= \epsilon_n
\end{aligned}
$$

Here the v's are residual vectors, the $\dot{B}_i$'s and $\ddot{B}_i$'s are the coefficient matrices of the parametric vectors $\dot{\delta}$ and $\ddot{\delta}_i$, respectively, and the ϵ's are the discrepancies between actual observations and their computed values. We note that, in accord with the above discussion, the $\dot{\delta}$ vector is distinguished by being common to all groups of the observational equations, whereas each $\ddot{\delta}_i$ vector appears in one and only one group of observational equations. The above system may be written

$$
(6)\qquad
\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}
+
\begin{bmatrix}
\dot{B}_1 & \ddot{B}_1 & 0 & \dots & 0 \\
\dot{B}_2 & 0 & \ddot{B}_2 & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
\dot{B}_n & 0 & 0 & \dots & \ddot{B}_n
\end{bmatrix}
\begin{bmatrix} \dot{\delta} \\ \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \vdots \\ \ddot{\delta}_n \end{bmatrix}
=
\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}
$$

or more compactly as,

(7) $\quad v + B\delta = \epsilon$.

If Λ denotes the covariance matrix of the vector of observations, the minimum variance adjustment is obtained by determining, of vectors v and δ that satisfy (7), that particular pair that leads to the minimization of the quadratic form

(8) $\quad s = v^T \Lambda^{-1} v$.

This process leads to the normal equations

(9) $\quad N\delta = c$,

where

(9a) $\quad N = B^T \Lambda^{-1} B$,

(9b) $\quad c = B^T \Lambda^{-1} \epsilon$.

We shall now assume that Λ is of the form

(10) $\quad \Lambda = \text{diag}\,(\Lambda_1 \; \Lambda_2 \; \cdots \; \Lambda_n)$

where each Λ_i consists to the covariance matrix of the observational vector corresponding to the residual vector v_i. Then inasmuch as B and ϵ are of the forms

(11)
$$B = \begin{bmatrix} \dot{B}_1 & \ddot{B}_1 & 0 & \dots & 0 \\ \dot{B}_2 & 0 & \ddot{B}_2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \dot{B}_n & 0 & 0 & \dots & \ddot{B}_n \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix},$$

It follows by direct evaluation of (9a) and (9b) that the normal equations (9) are of the form (4) in which

(12) $\quad \dot{N} = \dot{N}_1 + \dot{N}_2 + \dots + \dot{N}_n, \quad \dot{c} = \dot{c}_1 + \dot{c}_2 + \dots + \dot{c}_n$

wherein

(13) $\dot{N}_i = \dot{B}_i^T \Lambda_i^{-1} \dot{B}_i, \qquad \dot{c}_i = \dot{B}_i^T \Lambda_i^{-1} \epsilon_i$

and in which

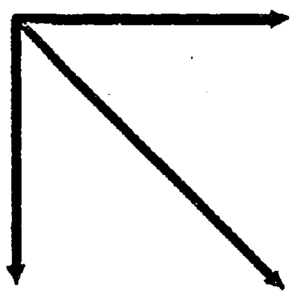
(14) $\bar{N}_i = \dot{B}_i^T \Lambda_i^{-1} \ddot{B}_i$

(15) $\ddot{N}_i = \ddot{B}_i^T \Lambda_i^{-1} \ddot{B}_i, \qquad \ddot{c}_i = \ddot{B}_i^T \Lambda_i^{-1} \epsilon_i .$

It is to be noted that this result applies to any set of observational equations of the form (5) for which the covariance matrix of the observational vector is of the form (10). Systems of this nature arise in a broad range of applications. In particular, such systems are generated in trajectory and orbital analysis when coefficients of tracking error models are to be recovered simultaneously with trajectory parameters. Before taking up specific examples, however, we shall continue in an abstract vein to study the properties of systems of normal equations having the particular patterned structure of (4).

Returning then, to equation (4), we first remark that when the number of elements in the $\ddot{\delta}$ vector is small compared with the number in the $\dot{\delta}$ vector, the block diagonality of the $\ddot{N}$ portion of the matrix is but of minor advantage to the solution of the system. However, in many problems the order of $\ddot{\delta}$ is much greater than that of $\dot{\delta}$, while at the same time the order of each $\ddot{\delta}_i$ is either smaller than that of $\dot{\delta}$ or else is of roughly comparable magnitude. Moreover, the number n of $\ddot{\delta}_i$ submatrices may be very large and without any particular preset limit. Here, the normal equations can assume enormous dimensions and the possibility of a practical solution hinges entirely on the block diagonality of $\ddot{N}$.

We shall call a system of normal equations of the form (4), a first order partitioned system and shall express the coefficient matrix schematically as

(16) $N \sim$ 

The sense of this diagram is the following: the horizontal and vertical arrows represent solid rows and columns, respectively, of elements that are not necessarily zero; the diagonal arrow represents a block diagonal array of submatrices; the blank regions between the arrows represent the portions of the matrix that are completely filled with zero elements. The arrowheads are indicative of the fact that the size of the matrix can grow without a preset limit. We shall find this diagrammatic representation of a first order partitioned system to be a convenient point of departure in our later discussions of higher order partitioned systems.

Aside from noting that the block diagonality of a first order partitioned system provides the key to its practical solution, we have yet to develop the details of the solution. We shall now remedy this deficiency. To begin, let us formally define the inverse of N to be

$$(17) \quad N^{-1} = M = \begin{bmatrix} \dot{M} & \bar{M} \\ \bar{M}^T & \ddot{M} \end{bmatrix}$$

where $\dot{M}$, $\bar{M}$, $\ddot{M}$ are each of the same order, respectively, as their counterparts $\dot{N}$, $\bar{N}$, $\ddot{N}$ in N. Inasmuch as NM = I, the unit matrix, by virtue of the definition of M, we may write the equivalent relation in partitioned form to get

$$(18) \quad \begin{bmatrix} \dot{N} & \bar{N} \\ \bar{N}^T & \ddot{N} \end{bmatrix} \begin{bmatrix} \dot{M} & \bar{M} \\ \bar{M}^T & \ddot{M} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} .$$

Performing the multiplication on the left hand side of the equation and equating the resulting elements with the corresponding elements on the right, we arrive at the following four matrix equations:

$$(19a) \quad \dot{N}\dot{M} + \bar{N}\bar{M}^T = I$$
$$(19b) \quad \dot{N}\bar{M} + \bar{N}\ddot{M} = 0$$
$$(19c) \quad \bar{N}^T\dot{M} + \ddot{N}\bar{M}^T = 0$$
$$(19d) \quad \bar{N}^T\bar{M} + \ddot{N}\ddot{M} = I \quad .$$

Since we have already exploited the fact that the inverse of a symmetric matrix is itself symmetric, the above system of four matrix equations involves but three unknown matrices: $\dot{M}$, $\bar{M}$, $\ddot{M}$. Various mutually consistent solutions are possible depending on which set of three one elects to solve. In view of the fact that we are ultimately going to regard $\ddot{N}$ as large, but block diagonal, it turns out appropriate to select equations (19a), (19c), and (19d) for the solution. Accordingly, first solving (19c) for $\bar{M}^T$ in terms of $\dot{M}$, we get

(20) $\quad \bar{M}^T = -\ddot{N}^{-1}\,\bar{N}^T\,\dot{M}$

and upon substituting this into (19a) and (19c), we have

(21a) $\quad \dot{N}\dot{M} - \bar{N}\ddot{N}^{-1}\bar{N}^T\dot{M} = I,$

(21b) $\quad -\bar{N}^T\dot{M}\bar{N}\ddot{N}^{-1} + \ddot{N}\ddot{M} = I.$

The solution of (21a) for $\dot{M}$ and that of (21b) for $\ddot{M}$ gives

(22a) $\quad \dot{M} = (\dot{N} - \bar{N}\ddot{N}^{-1}\bar{N}^T)^{-1}$

(22b) $\quad \ddot{M} = \ddot{N}^{-1} + \ddot{N}^{-1}\,\bar{N}^T\,\dot{M}\bar{N}\ddot{N}^{-1}\,.$

Once $\dot{M}$ has been obtained from (22a), the result can be substituted in (22b) to obtain $\ddot{M}$ and in (20) to obtain $\bar{M}^T$. Thereupon, the formal solution of the original normal equations (2) may be written

(23) $$\delta = \begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{N} & \bar{N} \\ \bar{N}^T & \ddot{N} \end{bmatrix}^{-1} \begin{bmatrix} \dot{c} \\ \ddot{c} \end{bmatrix} = \begin{bmatrix} \dot{M} & \bar{M} \\ \bar{M}^T & \ddot{M} \end{bmatrix} \begin{bmatrix} \dot{c} \\ \ddot{c} \end{bmatrix}$$

or as

(24a) $\quad \dot{\delta} = \dot{M}\dot{c} + \bar{M}\ddot{c},$

(24b) $\quad \ddot{\delta} = \bar{M}^T\dot{c} + \ddot{M}\ddot{c}.$

Alternatively, by virtue of the expression for $\bar{M}$ given by (20), $\dot{\delta}$ can be written as

(25) $\quad \dot{\delta} = \dot{M}(\dot{c} - \bar{N}\ddot{N}^{-1}\ddot{c}).$

Similarly, by virtue of (20) and (22b) the expression (24b) for $\ddot{\delta}$ becomes

$$(26) \qquad \ddot{\delta} = -\ddot{N}^{-1}\, \bar{N}^{T}\, \dot{M}\dot{c} + (\ddot{N}^{-1}\ddot{c} + \ddot{N}^{-1}\, \bar{N}^{T}\, \dot{M}\, \bar{N}\, \ddot{N}^{-1})\ddot{c}$$

The right hand side of which can be factored to yield

$$(27) \qquad \ddot{\delta} = \ddot{N}^{-1}\, \ddot{c} - \ddot{N}^{-1}\, \bar{N}^{T}\, \dot{M}(\dot{c} - \bar{N}\, \ddot{N}^{-1}\, \ddot{c}).$$

Equation (25) permits this to be written in the more compact form

$$(28) \qquad \ddot{\delta} = \ddot{N}^{-1}\, \ddot{c} - \ddot{N}^{-1}\, \bar{N}^{T}\, \dot{\delta}.$$

Equations (22a), (25) and (28) constitute the major preliminary results that we seek. They are preliminary because we have yet to exploit the block diagonality of $\ddot{N}$. This immediately permits us to write

$$(29) \qquad \ddot{N}^{-1} = \begin{bmatrix} \ddot{N}_1^{-1} & 0 & \dots & 0 \\ 0 & \ddot{N}_2^{-1} & \dots & 0 \\ \vdots & \vdots & \dots & 0 \\ 0 & 0 & \dots & \ddot{N}_n^{-1} \end{bmatrix}.$$

Since $\bar{N}$ is of the form

$$(30) \qquad \bar{N} = (\bar{N}_1\ \bar{N}_2\ \dots\ \bar{N}_n),$$

the expression $Q = \ddot{N}^{-1}\, \bar{N}^{T}$ becomes

$$(31) \qquad Q = \begin{bmatrix} \ddot{N}^{-1}\, \bar{N}^{T} \\ \ddot{N}_2^{-1}\, \bar{N}_2^{T} \\ \vdots \\ \ddot{N}_n^{-1}\, \bar{N}_n^{T} \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix}$$

and the expression $R = \bar{N}\, \ddot{N}^{-1}\, \bar{N}^{T} = \bar{N} Q$ becomes

$$(32) \qquad \begin{aligned} R &= \bar{N}_1\, \ddot{N}_1^{-1}\, \bar{N}_1^{T} + \bar{N}_2\, \ddot{N}_2^{-1}\, \bar{N}_2^{T} + \dots + \bar{N}_n\, \ddot{N}_n^{-1}\, \bar{N}_n^{T} \\ &= \bar{N}_1 Q_1 + \bar{N}_2 Q_2 + \dots + \bar{N}_n Q_n \\ &= R_1 + R_2 + \dots + R_n \end{aligned}$$

At this point we recall from equation (12) that the matrices $\dot{N}$ and $\dot{c}$ can be expressed as

(33a) $\dot{N} = \dot{N}_1 + \dot{N}_2 + \ldots + \dot{N}_n$

(33b) $\dot{c} = \dot{c}_1 + \dot{c}_2 + \ldots + \dot{c}_n$

in which $\dot{N}_i$, $\dot{c}_i$ are generated by the particular subset of observational equations containing $\ddot{\delta}_i$ (and hence by the same subset that generates $\bar{N}_i$, $\ddot{N}_i$, and $\ddot{c}_i$). It follows then, that $\dot{M}$ can be written as

(34) $\dot{M} = \dot{N}_1 + \dot{N}_2 + \ldots + \dot{N}_n - (R_1 + R_2 + \ldots + R_n)$

or as

(35) $\dot{M} = S_1 + S_2 + \ldots + S_n$

where

(36) $S_i = \dot{N}_i - R_i$.

Similarily, the expression $\dot{c} - \bar{N}\ddot{N}^{-1}\ddot{c}$ in (25) assumes the form

(37) $\dot{c} - \bar{N}\ddot{N}^{-1}\ddot{c} = (\dot{c}_1 + \dot{c}_2 + \ldots + \dot{c}_n) - (Q_1\ddot{c}_1 + Q_2\ddot{c}_2 + \ldots + Q_n\ddot{c}_n)$

$$= \bar{c}_1 + \bar{c}_2 + \ldots + \bar{c}_n$$

where

(38) $\bar{c}_i = \dot{c}_i - Q_i\ddot{c}_i$.

Proceeding further to examine the consequences of the block diagonality of the $\ddot{N}$ portion of the normal equations, we find that the expression (28) for $\ddot{\delta}$ can be written as

(39) $$\ddot{\delta} = \begin{bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \vdots \\ \ddot{\delta}_n \end{bmatrix} = \begin{bmatrix} \ddot{N}_1^{-1} & 0 & \ldots & 0 \\ 0 & \ddot{N}_2^{-1} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \ddot{N}_n^{-1} \end{bmatrix} \begin{bmatrix} \ddot{c}_1 \\ \ddot{c}_2 \\ \vdots \\ \ddot{c}_n \end{bmatrix} - \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix} \dot{\delta}$$

From this it follows that

$$(40)\qquad \ddot{\delta}_i = \ddot{N}_i^{-1}\,\ddot{c}_i - Q_i\,\dot{\delta},$$

which shows that each $\ddot{\delta}_i$ vector can be independently determined once the value of $\dot{\delta}$ has been obtained.

Inasmuch as the inverse of the coefficient matrix of the normal equations provides the covariance matrix of the adjusted parameters, it follows immediately from the foregoing development that

$$(41)\qquad \mathrm{var}\ (\dot{\delta}) = \dot{M}.$$

Since $\dot{M}$ has to be computed in order to obtain the solution for $\dot{\delta}$, the covariance matrix of $\dot{\delta}$ falls out as a direct by-product of the solution. A corresponding result does not hold for the vector $\ddot{\delta}$, for although the covariance matrix of $\ddot{\delta}$ is given by $\ddot{M}$, we found it possible to bypass the evaluation of $\ddot{M}$ in the computation of $\ddot{\delta}$ by equation (40). If, however, we return to equation (22b) and trace through the consequences of the block diagonality of $\ddot{N}$, we shall find that the diagonal submatrix of $\ddot{M}$ defining the covariance matrix of $\ddot{\delta}_i$ is given by

$$(42)\qquad \mathrm{var}\ (\ddot{\delta}_i) = \ddot{N}_i^{-1} + Q_i\,\dot{M}\,Q_i^T.$$

Furthermore, the covariance matrix of an arbitrary pair of vectors $\ddot{\delta}_i$, $\ddot{\delta}_j$ is given by

$$(43)\qquad \mathrm{cov}\ (\ddot{\delta}_i, \ddot{\delta}_j) = Q_i\,\dot{M}\,Q_j^T,\quad (i \neq j).$$

This completes the derivational development of the theory underlying first order patterned regression. Because the essential computational flow is perhaps obscured by derivational detail, it is appropriate that we conclude this section by extracting the essentials of the reduction. We shall take as our starting point the following set of four matrices concerned with the i th group of observational equations: $\dot{B}_i$, $\ddot{B}_i$, ϵ_i, Λ_i. Starting with i = 1, one first computes in terms of these matrices, the following five matrices:

(44) $\dot{N}_i = \dot{B}_i^T \Lambda_i^{-1} \dot{B}_i$

(45) $\bar{N}_i = \dot{B}_i^T \Lambda_i^{-1} \ddot{B}_i$

(46) $\ddot{N}_i = \ddot{B}_i^T \Lambda_i^{-1} \ddot{B}_i$

(48) $\dot{c}_i = \dot{B}_i^T \Lambda_i^{-1} \epsilon_i$

(49) $\ddot{c}_i = \ddot{B}_i^T \Lambda_i^{-1} \epsilon_i$

In terms of these the following four auxiliaries are computed:

(50) $Q_i = \ddot{N}_i^{-1} \bar{N}_i^T$

(51) $R_i = \bar{N}_i Q_i$

(52) $S_i = \dot{N}_i - R_i$

(53) $\bar{c}_i = \dot{c}_i - Q_i^T \ddot{c}_i$

As S_i and $\bar{c}_i$ are computed, they are added to the sum of their predecessors and only the cumulative sum is retained. After all groups of equations have been processed sequentially in this manner, one arrives at the final values.

(54) $S = S_1 + S_2 + \ldots + S_n,$

(55) $\bar{c} = \bar{c}_1 + \bar{c}_2 + \ldots + \bar{c}_n,$

The solution for $\dot{\delta}$ is then obtained immediately from

(56) $\dot{\delta} = \dot{M} c$

in which $\dot{M} = S^{-1}$ is also the covariance matrix of $\dot{\delta}$. Thereupon the solution for each $\ddot{\delta}_i$, in turn, is computed from

(57) $\ddot{\delta}_i = \ddot{N}_i^{-1} \ddot{c}_i - Q_i \dot{\delta}$,

parallel with which the covariance matrix is computed from

(58) $\mathrm{var}\,(\ddot{\delta}_i) = \ddot{N}_i^{-1} + Q_i \dot{M} Q_i^T.$

Finally, the observational residuals are obtained by substituting the values determined for $\dot{\delta}$ and $\ddot{\delta}_i$ into the original observational equations, which gives

$$(60) \qquad v_i = \epsilon_i - \dot{B}_i \dot{\delta} - \ddot{B}_i \ddot{\delta}_i .$$

The salient properties of the above solution of the normal equations characterizing first order patterned regression are

(1) The overall computational effort generally tends to increase only linearly with the number of parameters being recovered, rather than with the cube of the number as in adjustments where patterning is not or cannot be exploited. This renders practical the solution of important problems involving the simultaneous determination of thousands or tens of thousands of unknowns.

(2) The process of solving the normal equations proceeds apace with the formation of the normal equations Thus by the time the last group of observational equations has been processed, the great bulk of the computation required for the solution has also been completed.

(3) Core storage requirements are minimal in a computer program, since the processing for each of the basic groups of observational equations can proceed independently of that of the other groups.

Before taking up some of the applications of first order partitioned regression, we would note that the algorithm (equations (44) to (53)) leading to the computation of $S_i, \bar{c}_i$ can be put into the alternative, more compact form

$$(60a) \qquad \begin{bmatrix} \dot{N}_i & \bar{N}_i & \dot{c}_i \\ \bar{N}_i^T & \ddot{N}_i & \ddot{c}_i \end{bmatrix} = \begin{bmatrix} \dot{B}_i^T \\ \ddot{B}_i^T \end{bmatrix} \Lambda_{ii}^{-1} \begin{bmatrix} \dot{B}_i & \ddot{B}_i & \epsilon_i \end{bmatrix}$$

$$(60b) \qquad \begin{bmatrix} S_i & c_i \end{bmatrix} = \begin{bmatrix} \dot{N}_i & \dot{c}_i \end{bmatrix} - \begin{bmatrix} \bar{N}_i \end{bmatrix} \begin{bmatrix} \ddot{N}_i^{-1} \end{bmatrix} \begin{bmatrix} \bar{N}_i^T & \ddot{c}_i \end{bmatrix} .$$

This representation facilitates comparison with the algorithm to be developed for second order partitioned systems. The application of first order partitioned regression would have been sufficient for SAGA were it not for the fact that the program's general capabilities with respect to error modeling can lead to very large $\ddot{N}_i$ matrices.

4.2 Applications of First Order Partitioned Regression

The original application of the above development was made in Brown (1958a), where it provided the basis for the adjustment of a general photogrammetric net. Here, the $\dot{\delta}$ vector corresponded to the unknown elements of orientation of the cameras and each $\ddot{\delta}_i$ vector corresponded to the unknown X,Y,Z coordinates of a photographed point. One specialized application of this theory was to satellite geodesy (Brown, 1958b), wherein the $\dot{\delta}$ vector consisted of the unknown coordinates of a set of cameras and the $\ddot{\delta}_i$ vectors consisted of the unknown X,Y,Z coordinates of recorded flashes. This theory was successfully applied to the determination of the precise location of the location of Bermuda relative to the North American continent (Brown, 1959) and later to a detailed study of the feasibility of employing similar techniques on a larger scale to produce an ultra-precise survey of key tracking stations along the Atlantic Missile Range (Brown, 1960).

Another class of applications of first order partitioned regression emerged in 1959, when it was recognized that complex problems of trajectory analysis could be formulated in a manner leading to normal equations of essentially the same form as those successfully handled in the photogrammetric application. Here, the $\dot{\delta}$ vector consisted of unknown coefficients of tracking error models, whereas the $\ddot{\delta}_i$ consisted of unknown X,Y,Z coordinates of trajectory points. This application (Brown, et.al.1961) became known as EMBET (Error Model Best Estimate of Trajectory).

In due course it became recognized through application of EMBET to conventional tracking systems that certain systematic errors (e.g., zero set errors) could often be recovered more accurately through data reduction than they could be established by means of hardware. In 1960 Brown suggested that self-calibration by means of EMBET should be exploited as a guiding principle in the very design of new tracking systems. This philosophy was adopted in the design of the GLOTRAC system. Here, unknown reference frequencies of remote tracking stations were recovered in a specific EMBET reduction called GLAD (GLotrac ADjustment).

In Brown (1964b) and in Brown et. al. (1963), (1964), the feasibility of self-calibration of tracking systems by means of observations of satellites was extensively studied and simulated. This led, in particular, to an important version of EMBET called NEO-EMBET (N Epoch Orbital - Error Model Best Estimate of Trajectory) NEO-EMBET renders practical the simultaneous reduction of observations of an unlimited number of satellite arcs with all arcs being interrelated by certain common parameters (such as coordinates of tracking stations or stable coefficients of error models) but with each arc requiring the recovery of a fresh set of orbital elements and, in addition, with each pass observed by a given tracker requiring recovery of reinitialized error coefficients.

In a study performed for NASA (Brown, Stephenson, Hartwell, 1965) it was recommended that a special NEO-EMBET reduction be implemented in support of the GEOS I Short Arc Experiment. This recommendation was adopted by NASA and led to the development of an unusually powerful and flexible program called GDAP (GEOS Adjustment Program) which is discussed by Brown (1967a) and is described in detail by Lynn (1967). A significant application of GDAP to the establishment of a much improved survey of the GEOS short arc tracking network is reported by Brown (1968).

Under a recently completed contract with NASA, GDAP was employed as the starting point for the development of a still more advanced program called NAP (Network Analysis Program). Whereas, GDAP can accomodate an unlimited number of interrelated short arcs, NAP is able to accomodate an unlimited number of interrelated long arcs. However, the long arc application introduces what we shall presently take up as second order partitioned regression As has already been indicated, the purpose of the first part of this section is to provide the background needed for an understanding of second order partitioned regression Before taking up this topic, we would briefly cite five more instances where first order partitioned regression has been successfully applied.

In Brown (1964), the technique was applied to the reduction of stellar plates recorded by metric cameras of long focal length. Here, random errors in cataloged stellar positions become significant. Accordingly, the reduction was formulated so that parameters peculiar to the camera (focal length, distortion, orientation, etc.) were incorporated into the $\dot{\delta}$ vector, whereas the corrections to the cataloged positions of the ith star, were accounted for by $\ddot{\delta}_i$.

In Brown (1966) EMBET and NEO-EMBET reductions were developed and successfully applied to the reduction of Geodetic SECOR observations.

In Brown and Trotter (1967), first order partitioned regression provided the basis for the development of the Method of Continuous Traces, a technique for exploiting measurements of uninterrupted photographic traces of sun-illumined, passive satellites to establish precise geodetic positions. The technique does away with the need for synchronized chopping shutters, thus leading to less expensive, much simplified, and more reliable data acquisition.

In Brown (1967b) the development and successful application of a plate measuring comparator is described. The design of the comparator is based directly on principles of self-calibration as made practical through first order partitioned regression.

In Brown (1969) the calibration of metric cameras was approached from the standpoint that parameters of the inner cone (radial and decentering distortion, principal distance, and coordinates of principal point) are invariant over an indefinitely large number of exposures, whereas elements of exterior orientation may change from exposure to exposure. This formulation of the problem of camera calibration led to the formation of a first order partitioned system of normal equations.

From the foregoing discussions it is amply clear that the theory of first order partitioned regression has been put to extensive use during the course of the last decade. Undoubtedly many more applications will emerge in the next few years. There are, however, as in SAGA, significant problems leading to highly patterned normal equations that are not amenable to solution by the first order scheme. Many such problems fall into the province of second order partitioned regression which is the topic of the remainder of this section. We expect that during the next decade a host of applications of second order partitioned regression will emerge in various disciplines.

4.3 Second Order Partitioned Regression

As in our treatment of first order partitioned regression, we shall proceed first in an abstract vein in developing the theory of second order partitioned regression. We begin by postulating a patterned system of normal equations of the form

$$(61)\quad \begin{bmatrix} \hat{U} & \tilde{U}_1 & \tilde{U}_2 & \dots & \tilde{U}_n \\ \tilde{U}_1^T & \ddot{U}_1 & 0 & \dots & 0 \\ \tilde{U}_2^T & 0 & \ddot{U}_2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \tilde{U}_n^T & 0 & 0 & \dots & \ddot{U}_n \end{bmatrix} \begin{bmatrix} \hat{\delta} \\ \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{bmatrix} = \begin{bmatrix} \hat{c} \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

which, it will be noted, has the same structure as the first order system defined by equation (4). Also, as in (4), the index n can be indefinitely large. What distinguishes (61) as a second order system is its finer structure. The submatrices $\tilde{U}_i$, $\ddot{U}_i$, δ_i and c_i are of the form

$$(62a)\quad \tilde{U}_i = \begin{bmatrix} \dot{U}_i & \bar{U}_{i1} & \bar{U}_{i2} & \dots & \bar{U}_{in_i} \end{bmatrix}$$

$$(62b)\quad \ddot{U}_i = \begin{bmatrix} \dot{N}_i & \bar{N}_{i1} & \bar{N}_{i2} & \dots & \bar{N}_{in_i} \\ \bar{N}_{i1}^T & \ddot{N}_{i1} & 0 & \dots & 0 \\ \bar{N}_{i2}^T & 0 & \ddot{N}_{i2} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \bar{N}_{in_i}^T & 0 & 0 & \dots & \ddot{N}_{in_i} \end{bmatrix}, \quad \delta_i = \begin{bmatrix} \dot{\delta}_i \\ \ddot{\delta}_{i1} \\ \ddot{\delta}_{i2} \\ \vdots \\ \ddot{\delta}_{in_i} \end{bmatrix}, \quad c_i = \begin{bmatrix} \dot{c}_i \\ \ddot{c}_{i1} \\ \ddot{c}_{i2} \\ \vdots \\ \ddot{c}_{in_i} \end{bmatrix},$$

in which the index n_i can be indefinitely large. It is particularly to be noted that each $\ddot{U}_i$ has the structure of a first order system (and hence can grow without preset limit). This coupled with the fact that the number of $\ddot{U}_i$ is also without preset limit is the essence of a second order patterned system. Such systems may be represented diagrammatically as in

Figure 4. Here, the long horizontal and vertical arrows represent solid rows and columns of nonzero elements and the blocks arrayed along the diagonal individually consist of first order patterned systems. Thus a second order system may be described as a first order system of first order systems. Similarly, a third order system may be described as a first order system of second order systems. Although we shall not be directly concerned with the reduction of third and higher order systems, we have provided a diagrammatic representation of a third order system in Figure 5. While we have yet to find a solid, practical application for a third order scheme, there do exist, as we have already suggested, many significant applications of second order schemes.

Let us proceed formally with (61) as if it were a first order system. We could then follow the procedure of Section 1 to obtain a solution first for $\hat{\delta}$ and then for each of the $\ddot{\delta}_i$, in turn. The practical difficulty with this approach is that the $\ddot{U}_i$, unlike their counterparts $\ddot{N}_i$ in a first order scheme, may grow without set limit. While it is true that the required inverse of each $\ddot{U}_i$ could be computed from the algorithm developed for the first order scheme, the pivotal fact is that although $\ddot{U}_i$ is itself a sparse matrix, its inverse is not necessarily sparse, and indeed may be completely filled with nonzero elements. It follows that the formal solution of (61) for $\hat{\delta}$, given by the expression

$$(63)\quad \hat{\delta} = \left[\sum_{i=1}^{n} (\hat{U}_i - \tilde{U}_i \ddot{U}_i^{-1} \tilde{U}_i^T)\right]^{-1} \left[\sum_{i=1}^{n} (\hat{c}_i - \tilde{U}_i \ddot{U}_i^{-1} c_i)\right]$$

is computationally practical only when the dimensions of the $\ddot{U}_i$ are reasonably small. Accordingly, when the $\ddot{U}_i$ are considered to be indefinitely large, the above reduction is untenable, and the algorithm developed for a first order system cannot be applied directly to effect the reduction of a second order system. As we shall see, however, it can be applied in an indirect manner to produce a practical reduction.

Let us represent $\ddot{U}_i$ more compactly as

$$(64)\quad \ddot{U}_i = \begin{bmatrix} \dot{N}_i & \bar{N}_i \\ \tilde{N}_i^T & \dot{N}_i \end{bmatrix}$$

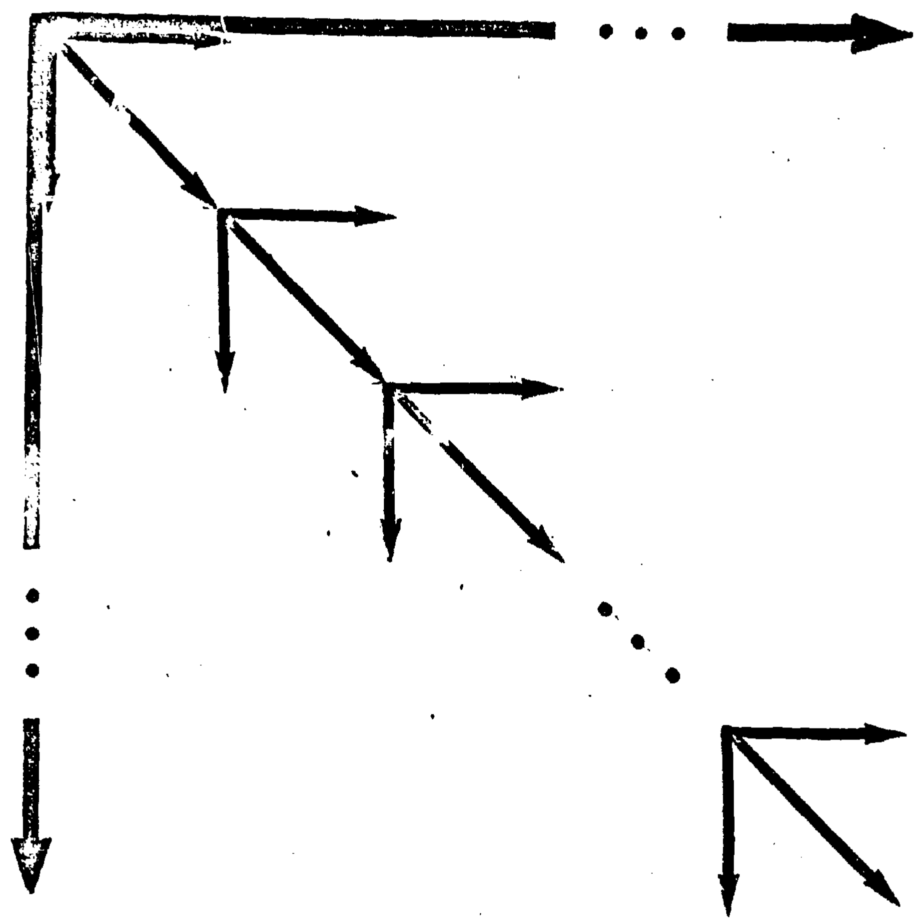


FIGURE 4. Schemctic of coefficient matrix characteristic of a second order partitioned system of normal equations.

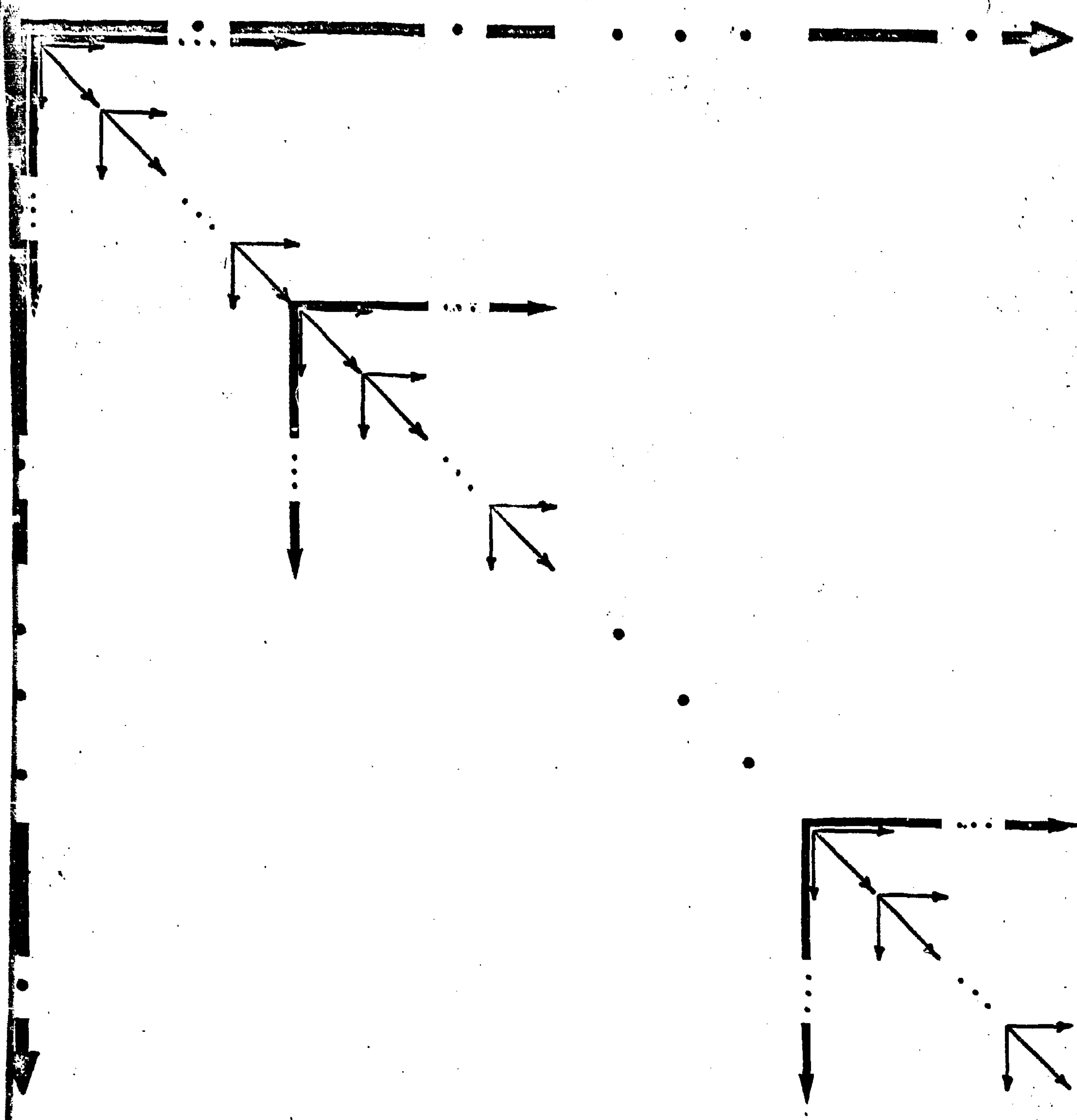


FIGURE 5. Schematic of coefficient matrix characteristic of a third order partitioned system of normal equations.

which corresponds to the partitioning in (62b). If we then designate the corresponding partitioning of $\widetilde{U}_i$, δ_i and c_i

$$\text{(65a)} \quad \widetilde{U}_i = (\dot{U}_i \;\; \bar{U}_i),$$

$$\text{(65b)} \quad \delta_i = \begin{bmatrix} \dot{\delta}_i \\ \ddot{\delta}_i \end{bmatrix},$$

$$\text{(65c)} \quad c_i = \begin{bmatrix} \dot{c}_i \\ \ddot{c}_i \end{bmatrix},$$

and also recognize that U and c can be expressed as sums of U_i and c_i, we can express equation (61) in greater detail as

$$\text{(66)} \quad \begin{bmatrix} \sum_{i=1}^{n} \hat{U}_i & \dot{U}_1 & \bar{U}_1 & \dot{U}_2 & \bar{U}_2 & \cdots & \dot{U}_n & \bar{U}_n \\ \dot{U}_1^T & \dot{N}_1 & \bar{N}_1 & 0 & 0 & \cdots & 0 & 0 \\ \bar{U}_1^T & \bar{N}_1^T & \ddot{N}_1 & 0 & 0 & \cdots & 0 & 0 \\ \dot{U}_2^T & 0 & 0 & \dot{N}_2 & \bar{N}_2 & \cdots & 0 & 0 \\ \bar{U}_2^T & 0 & 0 & \bar{N}_2^T & \ddot{N}_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \dot{U}_n^T & 0 & 0 & 0 & 0 & \cdots & \dot{N}_n & \bar{N}_n \\ \bar{U}_n^T & 0 & 0 & 0 & 0 & \cdots & \bar{N}_n^T & \ddot{N}_n \end{bmatrix} \begin{bmatrix} \hat{\delta} \\ \dot{\delta}_1 \\ \ddot{\delta}_1 \\ \dot{\delta}_2 \\ \ddot{\delta}_2 \\ \vdots \\ \dot{\delta}_n \\ \ddot{\delta}_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \hat{c}_i \\ \dot{c}_1 \\ \ddot{c}_1 \\ \dot{c}_2 \\ \ddot{c}_2 \\ \vdots \\ \dot{c}_n \\ \ddot{c}_n \end{bmatrix}$$

The expression $\widetilde{U}_i \, \ddot{U}_i^{-1} \, \widetilde{U}_i^T$ appearing in (63) can, by virtue of the above partitioning, be represented in expanded form as

$$\text{(67)} \quad \widetilde{U}_i \, \ddot{U}_i^{-1} \, \widetilde{U}_i^T = (\dot{U}_i \;\; \bar{U}_i) \begin{bmatrix} \dot{N}_i & \bar{N}_i \\ \bar{N}_i^T & \ddot{N}_i \end{bmatrix}^{-1} \begin{bmatrix} \dot{U}_i^T \\ \bar{U}_i^T \end{bmatrix}.$$

As in Section 1, we let

$$(68)\quad \begin{bmatrix} \dot{N}_i & \bar{N}_i \\ \bar{N}_i^T & \ddot{N}_i \end{bmatrix}^{-1} = \begin{bmatrix} \dot{M}_i & \bar{M}_i \\ \bar{M}_i^T & \ddot{M}_i \end{bmatrix}.$$

By virtue of the results of Section 1, a more detailed representation of the right hand side of (68) is

$$(69)\quad \begin{bmatrix} \dot{M}_i & \bar{M}_i \\ \bar{M}_i^T & \ddot{M}_i \end{bmatrix} = \begin{bmatrix} \dot{M}_i & -\dot{M}_i Q_i^T \\ -Q_i \dot{M}_i & \ddot{N}_i^{-1} + Q_i \dot{M}_i Q_i^T \end{bmatrix}$$

From (67), (68), (69) it follows that

$$(70)\quad \tilde{U}_i \ddot{U}_i^{-1} \tilde{U}_i^T = (\dot{U}_i \;\; \bar{U}_i) \begin{bmatrix} \dot{M}_i & -\dot{M}_i Q_i^T \\ -Q_i \dot{M}_i & \ddot{N}_i^{-1} + Q_i \dot{M}_i Q_i^T \end{bmatrix} \begin{bmatrix} \dot{U}_i^T \\ \bar{U}_i^T \end{bmatrix}$$

which reduces to

$$(71)\quad \tilde{U}_i \ddot{U}_i^{-1} \tilde{U}_i^T = \dot{U}_i \dot{M}_i \dot{U}_i^T - \dot{U}_i \dot{M}_i Q_i^T \bar{U}_i^T - \bar{U}_i Q_i \dot{M}_i \dot{U}_i^T + \bar{U}_i \ddot{N}_i^{-1} \bar{U}_i^T + \bar{U}_i Q_i \dot{M}_i Q_i^T \bar{U}_i^T$$

This may be written as

$$(72)\quad \tilde{U}_i \ddot{U}_i^{-1} \tilde{U}_i^T = \bar{U}_i \ddot{N}_i^{-1} \bar{U}_i^T + (\dot{U}_i - \bar{U}_i Q_i) \dot{M}_i (\dot{U}_i - \bar{U}_i Q_i)^T$$

In a similar manner, we find that the expression $\bar{U}_i \ddot{U}_i^{-1} \ddot{c}_i$ in (63) may be written as

$$(73)\quad \tilde{U}_i \ddot{U}_i^{-1} \ddot{c}_i = \bar{U}_i \ddot{N}_i^{-1} \ddot{c}_i + (\dot{U}_i - \bar{U}_i Q_i) \dot{M}_i (\dot{c}_i - Q_i^T \ddot{c}_i)$$

Although it is not yet readily apparent, equations (72) and (73) constitute the key results that make practical the reduction of a second order patterned scheme. This is because the evaluation of certain critical quantities in these expressions can be performed in terms of sums of matrices of low order that, in turn, can be readily evaluated during the course of the reduction of each of the first order systems implicit in the second order system. Before we demonstrate this, it is convenient to proceed as if the quantities required for the evaluation of (72) and (73) had been computed. It clarifies matters if we introduce the following set of auxiliaries

$$(74)\quad [\dot{N}]_i = \hat{U}_i - \bar{U}_i \ddot{N}_i^{-1} \bar{U}_i^T ,$$

$$(75)\quad [\bar{N}]_i = \dot{U}_i - \bar{U}_i Q_i ,$$

$$(76)\quad [\ddot{N}]_i^{-1} = \dot{M}_i = (\dot{N}_i - \bar{N}_i \ddot{N}_i^{-1} \bar{N}_i^T)^{-1}$$

$$(77)\quad [\dot{c}]_i = \hat{c}_i - \bar{U}_i \ddot{N}_i^{-1} \ddot{c}_i ,$$

$$(78)\quad [\ddot{c}]_i = \dot{c}_i - Q_i^T \ddot{c}_i = \bar{c}_i$$

As might be surmised, these auxiliaries have been formulated to be analogous to quantities involved in a first order process. The analogy is carried further with the formulation of the following set of secondary auxiliaries:

$$(79)\quad [Q]_i = [\ddot{N}]_i^{-1} [\bar{N}]_i^T$$

$$(80)\quad [R]_i = [\bar{N}]_i [Q]_i$$

$$(81)\quad [S]_i = [\dot{N}]_i - [R]_i$$

$$(82)\quad [\bar{c}]_i = [\dot{c}]_i - [Q]_i^T [\ddot{c}]_i$$

As the $[S]_i$ and $[c]_i$ are formed, they are added to the sum of their predecessors. This leads ultimately to the formation of the sums

$$(83)\quad [S] = [S]_1 + [S]_2 + \ldots + [S]_n ,$$

$$(84)\quad [c] = [\bar{c}]_1 + [\bar{c}]_2 + \ldots + [\bar{c}]_n .$$

But these are equivalent to the expressions in brackets on the right hand side of (63), a direct consequence of the fact that

$$(85)\quad [S]_i = \hat{U}_i - \tilde{U}_i \ddot{U}_i^{-1} \tilde{U}_i^T ,$$

$$(86)\quad [\bar{c}]_i = \hat{c}_i - \tilde{U}_i \ddot{U}_i^{-1} c_i .$$

Accordingly, the solution for $\hat{\delta}$ can be written

$$(87)\quad \hat{\delta} = [\dot{M}] [\bar{c}],$$

in which $[\dot{M}]$ is defined as

$$(88)\quad [\dot{M}] = [S]^{-1} .$$

Once $\hat{\delta}$ has been determined and eliminated, the second order scheme collapses to the independent reduction of n first order schemes. Specifically,

$$(89)\quad \dot{\delta}_i = \ddot{U}_i^{-1} \ddot{c}_i - [Q]_i \hat{\delta}$$

But by virtue of (62b) and (62c) the expression $\ddot{U}_i^{-1} c_i$ is the same as

$$(90)\quad \ddot{U}_i^{-1} c_i = \begin{bmatrix} \dot{N}_i & \bar{N}_{i1} & \bar{N}_{i2} \dots \bar{N}_{in_i} \\ \bar{N}_{i1}^T & \ddot{N}_{i1} & 0 \quad \dots 0 \\ \bar{N}_{i2}^T & 0 & \ddot{N}_{i2} \dots 0 \\ \vdots & \vdots & \vdots \qquad \vdots \\ \bar{N}_{in_i} & 0 & 0 \quad \dots \ddot{N}_{in_i} \end{bmatrix}^{-1} \begin{bmatrix} \dot{c}_i \\ \ddot{c}_{i1} \\ \ddot{c}_{i2} \\ \vdots \\ \ddot{c}_{in_i} \end{bmatrix}$$

the practical evaluation of which is precisely the problem considered in connection with first order partitioned regression. If we let the vector

$$(91)\quad \delta_i^0 = \begin{bmatrix} \dot{\delta}_i^0 \\ \ddot{\delta}_i^0 \end{bmatrix}$$

denote the intermediate or provisional solution obtained by applying the first order process to (90), the final solution may be expressed as

(92) $\delta_i = \delta_i^0 - [Q]_i \, \hat{\delta}$.

This shows that the reduction of a second order system may be developed in terms of independent reductions of n first order systems, the solutions of which are subject to subsequent modification once the vector $\hat{\delta}$ has been established.

The practicality of the above reduction hinges on the fact that the dimensions of the five basic auxiliaries $[\dot{N}]_i$, $[\bar{N}]_i$, $[\ddot{N}]_i^{-1}$, $[\dot{c}]_i$ and $[\ddot{c}]_i$ are determined by the orders of the vectors $\hat{\delta}$ and $\dot{\delta}_i$ which, unlike those of the vectors $\ddot{\delta}_i$, do not increase with n_i. Accordingly, these auxiliaries are not subject to unlimited growth as more and more data are processed. However, it remains to be shown that their computation is a practical matter, for this question had been bypassed in the development beginning with equation (74).

We begin by anticipating the result that the primary matrices $\hat{U}_i$, $\dot{U}_i$, $\dot{N}_i$, $\hat{c}_i$, $\dot{c}_i$ that appear explicitly in the second order system (66) are each expressible as sums of matrices generated at the level of the double subscripted matrices $\bar{N}_{ij}$, $\ddot{N}_{ij}$, $\ddot{c}_{ij}$. Thus we write

(93a) $\hat{U}_i = \hat{U}_{i1} + \hat{U}_{i2} + \ldots + \hat{U}_{in_i}$,

(93b) $\dot{U}_i = \dot{U}_{i1} + \dot{U}_{i2} + \ldots + \dot{U}_{in_i}$,

(93c) $\dot{N}_i = \dot{N}_{i1} + \dot{N}_{i2} + \ldots + \dot{N}_{in_i}$,

(93d) $\hat{c}_i = \hat{c}_{i1} + \hat{c}_{i2} + \ldots + \hat{c}_{in_i}$,

(93e) $\dot{c}_i = \dot{c}_{i1} + \dot{c}_{i2} + \ldots + \dot{c}_{in_i}$.

By virtue of the partitioning of $\bar{U}_i$ and $\ddot{N}_i$ implicit in (62a) and (62b), the expression $\ddot{N}_i^{-1} \, \bar{U}_i^T$ reduces to the form

(94) $\ddot{N}_i^{-1}\,\bar{U}_i^T = \begin{bmatrix} \dot{Q}_{i1} \\ \dot{Q}_{i2} \\ \vdots \\ \dot{Q}_{in_i} \end{bmatrix}$

in which $\dot{Q}_{ij}$ is defined by

(95) $\dot{Q}_{ij} = \ddot{N}_{ij}^{-1}\,\bar{U}_{ij}^T$.

If we then define the additional auxiliaries

(96) $\dot{R}_{ij} \quad = \bar{U}_{ij}\,\dot{Q}_{ij}$,

(97) $[\dot{N}]_{ij} = \hat{U}_{ij} - \dot{R}_{ij}$,

it follows that $[\dot{N}]_i$ can be expressed as

(98) $[\dot{N}]_i \quad = [\dot{N}]_{i1} + [\dot{N}]_{i2} + \ldots + [\dot{N}]_{in_i}$,

and similarly that $[\dot{c}]_i$ can be expressed as

(99) $[\dot{c}]_i \quad = [\dot{c}]_{i1} + [\dot{c}]_{i2} + \ldots + [\dot{c}]_{in_i}$

in which

(100) $[\dot{c}]_{ij} \quad = \hat{c}_{ij} - \dot{Q}_{ij}^T\,\ddot{c}_{ij}$.

Because $\bar{U}_i$ and Q_i are row and column vectors of $\bar{U}_{ij}$ and Q_{ij}, respectively, it further follows that $[\bar{N}]_i$ can be expressed as

(101) $[\bar{N}]_i \quad = [\bar{N}]_{i1} + [\bar{N}]_{i2} + \ldots + [\bar{N}]_{in_i}$

in which

(102) $[\bar{N}]_{ij} = \dot{U}_{ij} - \bar{U}_{ij}\,Q_{ij}$.

Finally, we note that the matrices $[\ddot{N}]_i^{-1} = \dot{M}_i$ and $[\ddot{c}]_i = \tau_i$ would be generated during the normal course of the reduction of a first order system. It follows, then, that by augmenting the reduction of a first order system by the generation of equations (95), (96), (97), (100) and (102) (all of which involve simple operations on matrices of low order) and by performing, in a cumulative manner, the summations (98), (99), (101), one can generate the auxiliaries (74) through (78) that are required for the reduction of the second order system.

We shall shortly extract from the above derivation a concise computational outline of the reduction of a second order system. But before so doing, we shall clarify certain points that might still be obscure. First, let us return to the matter of the evaluation in (63) of the expression $\tilde{U}_i \ddot{U}_i^{-1} \tilde{U}_i^T$ which we pointed out is impractical to accomplish in a direct manner when $\ddot{U}_i$ is large. The key to an appreciation of precisely why the indirect reduction just derived is practical, whereas a direct reduction is impractical, lies in a consideration of the final term on the right hand side of equation (72). This term can be written out more fully as

$$(103)\quad \bar{U}_i Q_i \dot{M}_i Q_i^T \bar{U}_i^T = \left[\bar{U}_{i1}\ \bar{U}_{i2} \ldots \bar{U}_{in_i}\right] \begin{bmatrix} Q_{i1} \\ Q_{i2} \\ \vdots \\ Q_{in_i} \end{bmatrix} \dot{M}_i \left[Q_{i1}^T\ Q_{i2}^T \ldots Q_{in_i}^T\right] \begin{bmatrix} \bar{U}_{i1}^T \\ \bar{U}_{i2}^T \\ \vdots \\ \bar{U}_{in_i}^T \end{bmatrix} .$$

The amount of computation required for the evaluation of this expression depends altogether on the order in which the multiplications are performed. An inefficient reduction would consist of evaluating first the expression $Q_i \dot{M}_i Q_i^T$ and then pre and post-multiplying this result by $\bar{U}_i$ and $\bar{U}_i^T$. An efficient reduction would consist of evaluating first the expression $\bar{U}_i Q_i$ and then pre and post-multiplying $\dot{M}_i$ by this result and its transpose. To see this readily, suppose that $\dot{M}_i$ and the $\bar{U}_{ij}$ and Q_{ij} were all scalars. Then one could easily verify that evaluation of (103) by the inefficient procedure would require on the order of $2n_i^2$ multiplications and n_i^2 additions, whereas evaluation by the efficient procedure would require about $2n_i$ multiplications and n_i additions. Accordingly, the practicability of the reduction of a second order system can be said to depend essentially on efficient manipulation of matrices. The direct evaluation of the key expression $\tilde{U}_i \ddot{U}_i^{-1} \tilde{U}_i^T$ is impractical simply because it entails matrix operations that are inefficient in view of the desired end result (note here that the full inverse of $\ddot{U}_i$ is not generally of interest, and hence its evaluation as an intermediary is of no direct value). By examining the finer structure of this expression, we find that means do exist for its efficient evaluation, and it is this fact that leads to a practical reduction of a second order system.

It will be noted that in developing the reduction of the second order system we adopted notation paralleling that used in the reduction of a first order system (compare equations (79)-(89) with (50)-(57)). This suggests that the reduction of a second order system is akin to the double application of the reduction of a first order system. That this is indeed so is especially obvious from the following alternative development attributable to John Stephenson (private communication). The original second order system (66) can be rearranged as follows

$$(104)\quad \left[\begin{array}{cccccc|cccc}
\sum_{i=1}^{n} \hat{U}_i & \dot{U}_1 & \dot{U}_2 & \dots & \dot{U}_n & & \bar{U}_1 & \bar{U}_2 & \dots & \bar{U}_n \\
\dot{U}_1^T & \dot{N}_1 & 0 & \dots & 0 & & \bar{N}_1 & 0 & \dots & 0 \\
\dot{U}_2^T & 0 & \dot{N}_2 & \dots & 0 & & 0 & \bar{N}_2 & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
\dot{U}_n^T & 0 & 0 & \dots & \dot{N}_n & & 0 & 0 & \dots & \bar{N}_n \\
\hline
\bar{U}_1^T & \bar{N}_1^T & 0 & \dots & 0 & & \ddot{N}_1 & 0 & \dots & 0 \\
\bar{U}_2^T & 0 & \bar{N}_2^T & \dots & 0 & & 0 & \ddot{N}_2 & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
\bar{U}_n^T & 0 & 0 & \dots & \bar{N}_n^T & & 0 & 0 & \dots & \ddot{N}_n
\end{array}\right]
\left[\begin{array}{c}
\hat{\delta} \\ \dot{\delta}_1 \\ \dot{\delta}_2 \\ \vdots \\ \dot{\delta}_n \\ \hline \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \vdots \\ \ddot{\delta}_n
\end{array}\right]
=
\left[\begin{array}{c}
\sum_{i=1}^{n} \hat{c}_i \\ \dot{c}_1 \\ \dot{c}_2 \\ \vdots \\ \dot{c}_n \\ \hline \ddot{c}_1 \\ \ddot{c}_2 \\ \vdots \\ \ddot{c}_n
\end{array}\right].$$

Let this be abbreviated as follows in accordance with the partitioning indicated by the broken lines:

$$(105)\quad \begin{bmatrix} \dot{P} & \bar{P} \\ \bar{P}^T & \ddot{P} \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{c} \\ \ddot{c} \end{bmatrix}$$

Then the reduced normal equations in the vector $\dot{\delta}$ as obtained by application of the first order algorithm for the elimination of $\ddot{\delta}$ is

$$(106)\quad (\dot{P} - \bar{P}\,\ddot{P}^{-1}\bar{P}^T)\,\dot{\delta} = \dot{c} - \bar{P}\,\ddot{P}^{-1}\ddot{c}.$$

It is readily verified that the expressions $\bar{P}\,\ddot{P}^{-1}\bar{P}^T$ and $\bar{P}\,\ddot{P}^{-1}\ddot{c}$ reduce to

(107)

$$\bar{P}\,\ddot{P}^{-1}\bar{P}^T = \begin{bmatrix} \sum_{i=1}^{n} \bar{U}_i\,\ddot{N}_i^{-1}\bar{U}_i^T & \bar{U}_1\,\ddot{N}_1^{-1}\bar{N}_1^T & \bar{U}_2\,\ddot{N}_2^{-1}\bar{N}_2^T & \ldots & \bar{U}_n\,\ddot{N}_n^{-1}\bar{N}_n^T \\ \bar{N}_1\,\ddot{N}_1^{-1}\bar{U}_1^T & \bar{N}_1\,\ddot{N}_1^{-1}\bar{N}_1^T & 0 & \ldots & 0 \\ \bar{N}_2\,\ddot{N}_2^{-1}\bar{U}_2^T & 0 & \bar{N}_2\,\ddot{N}_2^{-1}\bar{N}_2^T & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \bar{N}_n\,\ddot{N}_n^{-1}\bar{U}_n^T & 0 & 0 & \ldots & \bar{N}_n\,\ddot{N}_n^{-1}\bar{N}_n^T \end{bmatrix}, \quad \bar{P}\,\ddot{P}^{-1}\ddot{c} = \begin{bmatrix} \sum_{i=1}^{n} \bar{U}_i\,\ddot{N}_i^{-1}\ddot{c}_i \\ \bar{N}_1\,\ddot{N}_1^{-1}\ddot{c}_1 \\ \bar{N}_2\,\ddot{N}_2^{-1}\ddot{c}_2 \\ \vdots \\ \bar{N}_n\,\ddot{N}_n^{-1}\ddot{c}_n \end{bmatrix}.$$

The key points here are that $\bar{P}\,\ddot{P}^{-1}\bar{P}$ has precisely the same structure as $\dot{P}$ and that $\ddot{N}_i^{-1}$ is a block diagonal matrix. The latter fact permits the elements of the above matrices to be expressed as sums of low order matrices. Thus

$$(108)\quad \begin{aligned} \bar{U}_i\,\ddot{N}_i^{-1}\bar{U}_i^T &= \Sigma\,\bar{U}_{ij}\,\ddot{N}_{ij}^{-1}\,\bar{U}_{ij}^T \\ \bar{U}_i\,\ddot{N}_i^{-1}\bar{N}_i^T &= \Sigma\,\bar{U}_{ij}\,\ddot{N}_{ij}^{-1}\,\bar{N}_{ij}^T \\ \bar{N}_i\,\ddot{N}_i^{-1}\bar{N}_i^T &= \Sigma\,\bar{N}_{ij}\,\ddot{N}_{ij}^{-1}\,\bar{N}_{ij}^T \\ \bar{U}_i\,\ddot{N}_i^{-1}\ddot{c}_i &= \Sigma\,\bar{U}_{ij}\,\ddot{N}_{ij}^{-1}\,\ddot{c}_{ij} \\ \bar{N}_i\,\ddot{N}_i^{-1}\ddot{c}_i &= \Sigma\,\bar{N}_{ij}\,\ddot{N}_{ij}^{-1}\,\ddot{c}_{ij} \end{aligned}$$

where all the sums run from $j = 1$ to n_i. The form of the reduced normal equations (106) is

(109)

$$\begin{bmatrix} \sum_{i=1}^{n} (\hat{U}_i - \bar{U}_i \ddot{N}_i^{-1} \bar{U}_i^T) & \dot{U}_1 - \bar{U}_1 \ddot{N}_1^{-1} \bar{N}_1^T & \dot{U}_2 - \bar{U}_2 \ddot{N}_2^{-1} \bar{U}_2^T & \cdots & \dot{U}_n - \bar{U}_n \ddot{N}_n^{-1} \bar{U}_n^T \\ \dot{U}_1^T - \bar{N}_1 \ddot{N}_1^{-1} \bar{U}_1^T & \dot{N}_1 - \bar{N}_1 \ddot{N}_1^{-1} \bar{N}_1^T & 0 & & \\ \dot{U}_2^T - \bar{N}_2 \ddot{N}_2^{-1} \bar{U}_2^T & 0 & \dot{N}_2 - \bar{N}_2 \ddot{N}_2^{-1} \bar{N}_2^T & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \dot{U}_n^T - \bar{N}_n \ddot{N}_n^{-1} \bar{U}_n^T & 0 & 0 & \cdots & \dot{N}_n - \bar{N}_n \ddot{N}_n^{-1} \bar{N}_n^T \end{bmatrix} \begin{bmatrix} \hat{\delta} \\ \dot{\delta}_1 \\ \dot{\delta}_2 \\ \vdots \\ \dot{\delta}_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} (\hat{c}_i - \bar{U}_i \ddot{N}_i^{-1} \ddot{c}_i) \\ \ddot{c}_1 - \bar{N}_1 \ddot{N}_1^{-1} \ddot{c}_1 \\ \ddot{c}_2 - \bar{N}_2 \ddot{N}_2^{-1} \ddot{c}_2 \\ \vdots \\ \ddot{c}_n - \bar{N}_n \ddot{N}_n^{-1} \ddot{c}_n \end{bmatrix}$$

which in view of the auxiliaries (74) - (78) may be written more concisely as

$$\begin{bmatrix} \sum_{i=1}^{n} [\dot{N}]_i & [\bar{N}]_1 & [\bar{N}]_2 & \cdots & [\bar{N}]_n \\ [\bar{N}]_1^T & [\ddot{N}]_1 & 0 & \cdots & 0 \\ [\bar{N}]_2^T & 0 & [\ddot{N}]_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ [\bar{N}_n]^T & 0 & 0 & \cdots & [\ddot{N}]_n \end{bmatrix} \begin{bmatrix} \hat{\delta} \\ \dot{\delta}_1 \\ \dot{\delta}_2 \\ \vdots \\ \dot{\delta}_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} [\dot{c}]_i \\ [\ddot{c}]_1 \\ [\ddot{c}]_2 \\ \vdots \\ [\ddot{c}]_n \end{bmatrix} . \tag{110}$$

This reduced system is itself a first order system and its reduction proceeds in accordance with equations (79) - (89).

The above development makes it clear that the reduction of a second order system can be accomplished be a double application of the reduction of a first order system. All that remains to be done is to transcribe the reduction into a concise computational flow abstracted from derivational detail. To do this we begin with the fact that the system of observational equations that ultimately gives rise to the second order system is of the form

(111) $\quad v_{ij} + \hat{B}_{ij}\hat{\delta} + \dot{B}_{ij}\dot{\delta}_i + \ddot{B}_{ij}\ddot{\delta}_{ij} = \epsilon_{ij}$

$$i = 1,2,\ldots,n$$
$$j = 1,2,\ldots,n_i$$

The covariance matrix Λ associated with the observational vector is considered to be diagonal and partitioned as

(112) $\quad \Lambda = \text{diag}\,(\Lambda_1 \;\; \Lambda_2 \;\cdots\; \Lambda_n)$

where in turn

(113) $\quad \Lambda_i = \text{diag}\,(\Lambda_{i1} \;\; \Lambda_{i2} \cdots \Lambda_{in_i})$.

The algorithm for the reduction of a second order system then begins with the evalution of the following array of primary matrices

$$(114) \quad \begin{bmatrix} \hat{U}_{ij} & \dot{U}_{ij} & \bar{U}_{ij} & \hat{c}_{ij} \\ \dot{U}^T_{ij} & \dot{N}_{ij} & \bar{N}_{ij} & \dot{c}_{ij} \\ \bar{U}^T_{ij} & \bar{N}^T_{ij} & \ddot{N}_{ij} & \ddot{c}_{ij} \end{bmatrix} = \begin{bmatrix} \hat{B}^T_{ij} \\ \dot{B}^T_{ij} \\ \ddot{B}^T_{ij} \end{bmatrix} \Lambda^{-1}_{ij} \left[\hat{B}_{ij} \; \dot{B}_{ij} \; \ddot{B}_{ij} \; \epsilon_{ij}\right]$$

In terms of these the following primary reduced array is formed

$$(115) \quad \begin{bmatrix} [\dot{N}]_{ij} & [\bar{N}]_{ij} & [\dot{c}]_{ij} \\ [\bar{N}]^T_{ij} & [\ddot{N}]_{ij} & [\ddot{c}]_{ij} \end{bmatrix} = \begin{bmatrix} \hat{U}_{ij} & \dot{U}_{ij} & \hat{c}_{ij} \\ \bar{U}^T_{ij} & \dot{N}_{ij} & \dot{c}_{ij} \end{bmatrix} - \begin{bmatrix} \bar{U}_{ij} \\ \bar{N}_{ij} \end{bmatrix} \ddot{N}^{-1}_{ij} \left[\bar{U}^T_{ij} \;\; \bar{N}^T_{ij} \;\; \ddot{c}_{ij}\right]$$

As each such reduced array is formed it is added to the sum of its predecessors thereby producing, after running through $j = 1,2,\ldots,n_i$,

$$(116) \quad \begin{bmatrix} [\dot{N}]_i & [\bar{N}]_i & [\dot{c}]_i \\ [\bar{N}]^T_i & [\ddot{N}]_i & [\ddot{c}]_i \end{bmatrix} = \begin{bmatrix} \Sigma[\dot{N}]_{ij} & \Sigma[\bar{N}]_{ij} & \Sigma[\dot{c}]_{ij} \\ \Sigma[\bar{N}]_{ij} & \Sigma[\ddot{N}]_{ij} & \Sigma[\ddot{c}]_{ij} \end{bmatrix}.$$

The provisional solution for $\dot{\delta}_i$ is obtained from this array by means of

$$(117)\quad \dot{\delta}_i^{o} = [\dot{N}]_i^{-1}\,[\dot{c}]_i$$

As each array of the form (116), is formed its elements are operated on to form the secondary reduced array

$$(118)\quad \Big[[S]_i\ [\bar{c}]_i\Big] = \Big[[\dot{N}]_i\ [\dot{c}]_i\Big] - [\bar{N}]_i\,[\dot{N}]_i^{-1}\Big[[\bar{N}]_i^T\ [\dot{c}]_i\Big]\ .$$

These are then summed as they formed to produce, after i has run from 1 to n,

$$(119)\quad \Big[[S]\ [\bar{c}]\Big] = \left[\sum_{i=1}^{n}[S]_i\ \ \sum_{i=1}^{n}[c]_i\right].$$

In terms of these, the solution for $\hat{\delta}$ is

$$(120)\quad \hat{\delta} = [S]^{-1}\,[\bar{c}].$$

The revised solutions for each of the $\dot{\delta}_i$ are then obtained from

$$(121)\quad \dot{\delta}_i = \dot{\delta}_i^{o} - [\dot{N}]_i^{-1}\,[\bar{N}]_i^T\,\hat{\delta} \qquad i = 1,2,\ldots,n$$

and in terms of these the solutions for each of the $\ddot{\delta}_{ij}$ are obtained from

$$(122)\quad \ddot{\delta}_{ij} = \ddot{N}_{ij}^{-1}\,\ddot{c}_{ij} - \ddot{N}_{ij}^{-1}\,\bar{N}_{ij}^T\,\dot{\delta}_i\ .$$

With $\hat{\delta}$, $\dot{\delta}_i$ and $\ddot{\delta}_{ij}$ thus determined, the residuals may be computed from the observational equations (111)

$$(123)\quad v_{ij} = \epsilon_{ij} - (\hat{B}_{ij}\hat{\delta} + \dot{B}_{ij}\dot{\delta}_i + \ddot{B}_{ij}\ddot{\delta}_{ij})$$

Alternatively, if the reduction is iterated until the corrections to the parameters become insignificant, the residuals may be computed from

$$(124)\quad v_{ij} = \epsilon_{ij}$$

where ϵ_{ij} now denotes the discrepancy vector arising from the substitution of the final values of the parameters into the original observational equations.

The error propagation associated with the adjustment requires the computation of those diagonal blocks of the inverse of the coefficient matrix that correspond to the vectors $\hat{\delta}$, $\dot{\delta}_i$ and $\ddot{\delta}_{ij}$. For $\hat{\delta}$ we have immediately

(125) $\quad \mathrm{var}(\hat{\delta}) = [S]^{-1} = [\dot{M}]$

For $\dot{\delta}_i$, the appropriate result is

(126) $\quad \mathrm{var}(\dot{\delta}_i) = [S]_i^{-1} + [\dot{Q}]_i\,[\dot{M}]\,[\dot{Q}]_i^T$

where $[\dot{Q}]_i = [\dot{N}]_i^{-1}\,[\bar{N}]_i^T$. Finally, that for $\ddot{\delta}_{ij}$ is given by

(127) $\quad \mathrm{var}(\ddot{\delta}_{ij}) = \ddot{N}_{ij}^{-1} + Q_{ij}\,\mathrm{var}(\dot{\delta}_i)\,Q_{ij}^T$,

where $Q_{ij} = \ddot{N}_{ij}^{-1}\,\bar{N}_{ij}$.

Equations (114) through (127) provide in concise form the complete reduction of a second order system. Such a system can become indefinitely large in either or both of two ways: namely, n can become indefinitely large (as in a first order system) or some or all of the n_i can become indefinitely large. When the reduction of a second order system is accomplished by means of the above algorithm, the overall computational effort tends to increase linearly with the dimensions of both n and n_i. Unless the dimensions of the $\dot{N}_i$ and $\ddot{N}_{ij}$ matrices are grossly disparate, the precise manner in which the normal equations grow is of little computational consequence. To conclude this section we would note that the three points listed at the end of Section 4.1 as characterizing the reduction of a first order system apply equally well to a second order system.

4.4 Applications of Second Order Partitioned Regression

Although the programs SAGA and GDAP are both short arc reductions, they do not apply the same approach to the solution of the normal equations. In GDAP, the algorithm for a first order patterned system is employed. This is practical because the

$\ddot{U}_i$ characteristically generated in GDAP reductions do not become excessively large. Thus relatively little is to be gained in GDAP from application of the second order algorithm. In SAGA, on the other hand, the $\ddot{U}_i$ can become quite large, particularly when error models are reinitialized several times over a pass at many of the participating stations. For this reason, SAGA employs the algorithm for second order systems, and the program's capabilities are thus considerably expanded over what would have otherwise been practical.

5.0 AUTOREGRESSIVE FEEDBACK

5.1 Introduction

SAGA has an option designed to take serially correlated errors into account. This capability is of particular value when high sampling rates are exercised, as in optical chopping of passive satellites. Here, successive observations are likely to inherit common, slowly changing errors by virtue of their proximity in time and space. Even though such errors may be well submerged in the noise, they ultimately set limits to attainable accuracies — limits that cannot be overcome by increased sampling. Thus, we can assert that in optical chopping, a set of 500 points on a given plate is, by virtue of low order serial correlation, unlikely to contain significantly more information than a representative sample of 50 points. Indeed it can be said in general that, even with systematic errors completely removed, errors in most channels of tracking observations are not likely to be strictly Gaussian in character, but, rather, are likely to be serially correlated to a greater or lesser degree. In Brown, Bush, Sibol (1963) we suggested that the key to the practical resolution of such difficulties might well lie in application of a basic result of autoregressive theory derived by Wise in 1955 ("The Autocorrelation Function and the Spectral Density Function," Biometrika 42, pp. 151-159). In Brown, Bush, Sibol (1964) we enlarged on this theme and developed the essence of what we now call 'autoregressive feedback.' We have since implemented a limited version of this concept in a successful application to the reduction of geodetic SECOR observations, Brown (1966). Not only does autoregressive feedback appear to provide a practical answer to the problem of serially correlated errors, it also appears to be an excellent means for taking into account the effects of unmodelled systematic error which induce serial correlation into observational residuals resulting from a conventional adjustment. We consider autoregressive feedback to be an exciting development, particularly since, as will be seen, its implementation can be so readily accomplished as a natural adjunct to conventional adjustments which ignore the presence of serial correlation Its incorporation into SAGA provides the program with a capability that, we feel, will become of increasing importance as the significance of serial correlation becomes more generally appreciated.

5.2 The Autoregressive Model

A stationary sequence of serially correlated errors $\epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \ldots$ is governed by an autoregressive process in which

$$(1) \qquad \epsilon_t = \alpha_1 \epsilon_{t-1} + \alpha_2 \epsilon_{t-2} + \ldots + \alpha_p \epsilon_{t-p} + \eta_t$$

where the α's are constant coefficients and η_t is a random impulse of zero mean and variance σ^2. According to this process, the i th error in the sequence is generated as a fixed linear combination of a set of its predecessors in combination with a superimposed, strictly random impulse. The process indicated in equation (1) is said to be of pth order because p coefficients are involved in its description. Specific examples of the character of errors generated by various first order autoregressive processes are to be found in Brown, Bush, Sibol (1963). Experience thus far indicates that autoregressive processes of low order provide satisfactory stochastic models for errors encountered in channels of tracking observations. In the case of Geodetic SECOR, sampled at one point per ten seconds, a first order process (i.e., $\epsilon_t = \alpha_1 \epsilon_{t-1} + \eta_t$) has been found generally to provide a satisfactory description of the error process (Brown (1966)). In Section 5.4 we shall provide statistical criteria for what constitutes a 'satisfactory' autoregressive model.

5.3 Inverse Covariance Matrix of Autoregressive Process

As we pointed out in our earlier work, the practical utility of the autoregressive process in the adjustment of tracking observations stems from the fact that, for a given set of autoregressive coefficients (which, as we shall see, can be estimated from the observations), one can compute the inverse of the covariance matrix of the observational channel analytically. Equally important, the inverse turns out to be a multidiagonal matrix, the number of diagonals being equal to 2p+1 for a process of pth order. In Brown, Bush, Sibol (1964) we showed that the basic result to this effect derived originally by Wise could be put into a more convenient form. Namely, if Λ denotes the covariance matrix of an autoregressive process governed by coefficients $\alpha_1, \alpha_2, \ldots, \alpha_p$, then Λ^{-1} can be expressed as the following product of lower and upper triangular matrices:

$$
(2) \qquad \sigma^2 \Lambda^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ \alpha_1 & -1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & \alpha_1 & -1 & 0 & \cdots & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \alpha_{n-4} & \cdots & -1 \end{bmatrix} \begin{bmatrix} -1 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} \\ 0 & -1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-2} \\ 0 & 0 & -1 & \alpha_1 & \cdots & \alpha_{n-3} \\ 0 & 0 & 0 & -1 & \cdots & \alpha_{n-4} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{bmatrix}
$$

in which $\alpha_n = 0$ for $n > p$. If this result is applied, for instance, to a third order process involving coefficients $\alpha_1, \alpha_2, \alpha_3$, one finds, upon performing the above matrix multiplication that Λ^{-1} assumes the form

$$
(3) \quad \sigma^2 \Lambda^{-1} = \begin{bmatrix} a_{-3} & b_{-2} & c_{-1} & d & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_{-2} & a_{-2} & b_{-1} & c & d & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c_{-1} & b_{-1} & a_{-1} & b & c & d & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ d & c & b & a & b & c & d & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & d & c & b & a & b & c & d & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & d & c & b & a & b & c & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & b & a & b & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & c & b & a & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a & b & c & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & b & a_{-1} & b_{-1} & c_{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c & b_{-1} & a_{-2} & b_{-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & d & c_{-1} & b_{-2} & a_{-3} \end{bmatrix}
$$

in which

$$
(4)\quad
\begin{array}{llll}
a = 1 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 & a_{-1} = 1 + \alpha_1^2 + \alpha_2^2 & a_{-2} = 1 + \alpha_1^2 & a_{-3} = 1 \\
b = -\alpha_1 + \alpha_1\alpha_2 + \alpha_2\alpha_3 & b_{-1} = -\alpha_1 + \alpha_1\alpha_2 & b_{-2} = -\alpha_1 & \\
c = -\alpha_2 + \alpha_2\alpha_3 & c_{-1} = -\alpha_2 & & \\
d = -\alpha_3 & & &
\end{array}
$$

This illustration is sufficiently general to demonstrate certain properties of the inverse that are of pivotal importance to our concept of autoregressive feedback:

(1) the inverse is multi-diagonal, the number of diagonals being equal to 7 for p=3 (or equal to 2p+1 in general)

(2) except for pxp submatrices comprising the upper and lower corners of the matrix, the elements of each diagonal are constant, the number of different constants being equal to p, the order of the process

(3) the formulas for the elements of the matrix are subject to a systematic, easily generalized development.

Expressions for second and first order processes can be obtained from the above development by successively equating α_3 and α_2 equal to zero in eqs. (4).

5.4 Estimation of Autoregressive Coefficients

Let us assume that by some means one has obtained a vector of residuals $(v_1, v_2, \ldots, v_n)$ that constitutes an estimate of the vector of actual errors $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$. One can then generate a set of autocorrelation coefficients $\rho_1, \rho_2, \ldots$ in the usual manner from

$$
(5)\quad \rho_k = \frac{\Sigma\, v_i\, v_{i-k}}{\Sigma\, v_i^2}
$$

If we now write the autoregressive function in terms of residuals

(6) $$v_t = \alpha_1 v_{t-1} + \alpha_2 v_{t-2} + \ldots + \alpha_p v_{t-p} + \eta_t$$

and regard this as defining a set of observational equations involving the autoregressive coefficients as unknowns, we can employ a least squares adjustment to generate a system of normal equations for the determination of the α's. These turn out to assume the form

(7) $$\begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \ldots & \rho_p \\ \rho_1 & 1 & \rho_1 & \rho_2 & \ldots & \rho_{p-1} \\ \rho_2 & \rho_1 & 1 & \rho_1 & \ldots & \rho_{p-2} \\ \rho_3 & \rho_2 & \rho_1 & 1 & \ldots & \rho_{p-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \rho_p & \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & \ldots & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \vdots \\ \rho_p \end{bmatrix} .$$

In practice, one would wish to establish the autoregressive process of lowest order that satisfactorily models the observed process. To do this one would begin with a first order process which, from (7), would lead to the estimate

(8) $$\alpha_1 = \rho_1 .$$

From this one would compute the secondary residuals η_t from

(9) $$\eta_t = \epsilon_t - \alpha_1 \epsilon_{t-1} .$$

If these secondary residuals turn out to be serially uncorrelated, the first order process can be accepted as satisfactory. To determine whether or not the η's are independent, one would first compute their first order coefficient of correlation from

$$(10) \quad r = \frac{\Sigma \eta_i \eta_{i-1}}{\Sigma \eta_i^2} .$$

To determine whether or not r is significantly different from zero, one would employ the well known result that the quantity

$$(11) \quad z = \tfrac{1}{2} \ln \frac{1+r}{1-r}$$

is approximately normally distributed with mean and variance of

$$(12) \quad \mu_z = \tfrac{1}{2} \ln \frac{1+\rho}{1-\rho}$$

$$(13) \quad \sigma_z^2 = 1/(n-3)$$

in which ρ is the true, but unknown, correlation coefficient. It follows that, at the 95% level of confidence, r will differ significantly from zero only if

$$(14) \quad z > 1.96/\sqrt{n-3} .$$

If r should turn out to be significantly different from zero, one would then try a second order process. Here, according to (7), the coefficients would be established from

$$(15)\qquad \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}$$

which has the solution

$$(16)\qquad \begin{aligned} \alpha_1 &= (\rho_1 - \rho_1\,\rho_2)/(1-\rho_1^2), \\ \alpha_2 &= (-\rho_1^2 + \rho_2)/(1-\rho_1^2). \end{aligned}$$

The secondary residuals for the second order process are

$$(17)\qquad \eta_i = \epsilon_i - (\alpha_1 \epsilon_{i-1} + \alpha_2 \epsilon_{i-2}).$$

One would test these for serial independence in precisely the same manner as described above for the first order process.

The above procedure would be repeated until a satisfactory fit is obtained. Surprisingly, the simple first order process has been found to be altogether sufficient for ranging residuals in most cases we have so far examined. Once a satisfactory autoregressive model has been established, the spectral density function can be computed analytically from the following elegant result derived by Wise:

$$(18)\qquad \nu(\theta) = \sigma^2/(-1 + \alpha_1 e^{i\theta} + \alpha_2 e^{i2\theta} + \ldots + \alpha_p e^{ip\theta})(-1 + \alpha_1 e^{-i\theta} + \alpha_2 e^{-i2\theta} + \ldots + \alpha_p e^{-ip\theta})$$

where $i = \sqrt{-1}$ and θ assumes the discrete sample values

$$(19)\qquad \theta = \frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \ldots, \frac{2(n-1)}{n}\pi, 2\pi .$$

If Δt denotes the time between successive points and $T = n\,\Delta t$ denotes the total time span, θ may be put in the form

$$(20)\quad \theta_j = \frac{2j\Delta t}{T}\pi, \qquad j = 1, 2, \ldots, n$$

where θ_j may be said to correspond to the frequency of $1/2j\,\Delta t$ cycles per second. For the first order process, eq. (18) reduces to the following well known expression for the spectral density function of the damped exponential autocorrelation function:

$$(21)\quad \nu(\theta) = \sigma^2/\,(1 - 2\rho\cos\theta + \rho^2)$$

in which we have set $\alpha_1 = \rho$.

5.5 Refined Normal Equations (Single Channel Case)

From the foregoing it can be seen that a successful autoregressive analysis of residuals provides the solution to two central problems of random error analysis: (1) the determination of the inverse covariance matrix of the observational vector, (2) the determination of the spectral density function of the error process. It remains to be shown precisely how this information can be used in a refinement of the adjustment and what its use entails in the way of additional complication. For this purpose we shall consider the specific case of an adjustment of a single channel of observations governed by a first order autoregressive process. This is sufficient to demonstrate the general principles of the operation. Accordingly, we let

$$(22)\quad \epsilon_i = \rho\,\epsilon_{i-1} + \eta_i$$

define a first order process in which

(23) $E(\eta_t) = 0$

(24) $\text{var}(\eta_t) = \sigma^2$ (variance of high frequency component of noise).

(25) $\text{cov}(\eta_t, \eta_{t-k}) = 0$ for all $k > 0$

Then it is easily shown that

(26) $E(\epsilon_t) = 0$,

(27) $\text{var}(\epsilon_t) = \sigma^2/(1-\rho^2) =$ total error variance,

(28) $\text{cov}(\epsilon_t, \epsilon_{t-k}) = \dfrac{\rho^k \sigma^2}{1-\rho^2}$,

(29) $\text{cor}(\epsilon_t, \epsilon_{t-k}) = \rho^k$.

It follows that the covariance matrix Λ of an n vector of errors is

$$(30)\quad \Lambda = \frac{\sigma^2}{1-\rho^2}\begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^n \\ \rho & 1 & \rho & \cdots & \rho^{n-1} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \rho^n & \rho^{n-1} & \rho^{n-2} & \cdots & 1 \end{bmatrix}$$

Let us now assume that the observational vector is employed in an adjustment. Then the normal equations can be expressed as

$$(31)\quad (B^T \Lambda^{-1} B)\delta = B^T \Lambda^{-1} \epsilon$$

in which δ denotes the vector of parametric corrections and the matrix B and the vector ϵ can be decomposed into

$$(32)\quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix},$$

where the b's are row vectors of order equal to that of δ and the ϵ's are scalars. In a conventional adjustment the covariance matrix is taken to be diagonal (ρ=0), and its inversion is, therefore, trivial. In the present instance, however, the covariance matrix is completely filled (eq.(30)), a fact which introduces complications. Because Λ is in this instance generated by a first order autoregressive process, we may employ the results of 5.3 to write immediately the following expression for Λ^{-1}

$$(33)\quad \Lambda^{-1} = \frac{1}{\sigma^2} \begin{bmatrix} 1 & -\rho & 0 & 0 & \dots & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 & \dots & 0 \\ 0 & -\rho & 1+\rho^2 & -\rho & \dots & 0 \\ 0 & 0 & -\rho & 1+\rho^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

We shall now trace through the consequences of the patterned and regular structure of Λ^{-1}. First we note that Λ^{-1} can be decomposed as follows:

(34) $$\sigma^2\Lambda^{-1} = (1+\rho^2)I - \rho U - \rho U^T - \rho^2 \Delta$$

In which I is an nxn unit matrix and U and Δ are nxn matrices of the form

(35) $$U = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

If we now let N denote the coefficient matrix of the normal equations, we may write

(36) $$N = B^T \Lambda^{-1} B$$

$$= \frac{1}{\sigma^2} (b_1^T \; b_2^T \; \dots \; b_n^T) \left[(1+\rho^2) I - \rho U - \rho U^T - \rho^2 \Delta\right] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

This expands to

$$(37)\quad N = \frac{1}{\sigma^2}\Big[(1+\rho^2)(b_1^T b_1 + b_2^T b_2 + \ldots + b_n^T b_n)$$

$$-\rho\ (b_1^T b_2 + b_2^T b_3 + \ldots + b_{n-1}^T b_n)$$

$$-\rho\ (b_2^T b_1 + b_3^T b_2 + \ldots + b_n^T b_{n-1})$$

$$-\rho^2\ (b_1^T b_1 + b_n^T b_n)\Big]$$

If we define

$$(38)\quad \Delta b_i = b_{i+1} - b_i$$

and note that then

$$(39)\quad b_{i-1}^T b_i = b_{i-1}^T (b_{i-1} + \Delta b_{i-1}) = b_{i-1}^T b_{i-1} + b_{i-1}^T \Delta b_{i-1}\ ,$$

$$(40)\quad b_i^T b_{i-1} = b_i^T (b_i - \Delta b_{i-1}) = b_i^T b_i - b_i^T \Delta b_{i-1}\ ,$$

it follows that

$$(41)\quad b_{i-1}^T b_i + b_i^T b_{i-1} = b_{i-1}^T b_{i-1} + b_i^T b_i + (b_{i-1}^T - b_i^T)\,\Delta b_{i-1}$$

$$= b_{i-1}^T b_{i-1} + b_i^T b_i + \Delta b_{i-1}^T\, \Delta b_{i-1}\ .$$

When this result is substituted into (37) and appropriate algebraic manipulations are performed, one obtains

$$(42)\quad N = \frac{1}{\sigma^2}\Big[(1-2\rho+\rho^2)(b_1^T b_1 + b_2^T b_2 + \ldots + b_n^T b_n)$$

$$+\rho\ (\Delta b_1^T \Delta b_1 + \Delta b_2^T \Delta b_2 + \ldots + \Delta b_{n-1}^T\ \Delta b_{n-1})$$

$$+\rho\ (1-\rho)(b_1^T b_1 + b_n^T b_n)\Big]$$

which is the same as

$$(43)\quad N = \frac{1}{\sigma^2}\Big[(1-\rho)^2\ B^T B + \rho\, \Delta B^T\, \Delta B + \rho(1-\rho)(b_1^T b_1 + b_n^T b_n)\Big]$$

in which

$$(44)\quad \Delta B = \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \vdots \\ \Delta b_{n-1} \end{bmatrix} = \begin{bmatrix} b_2 - b_1 \\ b_3 - b_2 \\ \vdots \\ b_n - b_{n-1} \end{bmatrix} .$$

In an analogous manner one obtains for the right hand side of the normal equations

$$(45)\quad c = B^T \Lambda^{-1} \epsilon$$

$$= \frac{1}{\sigma^2} \left[(1-\rho^2)\, B^T \epsilon + \rho \Delta B^T \Delta\epsilon + \rho(1-\rho)(b_1^T \epsilon_1 + b_n^T \epsilon_n) \right]$$

in which

$$(46)\quad \Delta\epsilon = \begin{bmatrix} \Delta\epsilon_1 \\ \Delta\epsilon_2 \\ \\ \Delta\epsilon_{n-1} \end{bmatrix} = \begin{bmatrix} \epsilon_2 - \epsilon_1 \\ \epsilon_3 - \epsilon_2 \\ \\ \epsilon_n - \epsilon_{n-1} \end{bmatrix} .$$

Equations (43) and (45) are the results sought. In analyzing them we first note that when $\rho = 0$, the normal equations reduce to the usual expression:

$$(47)\quad \frac{1}{\sigma^2} (B^T B)\, \delta = \frac{1}{\sigma^2} B^T \epsilon \; .$$

Next we note that since ΔB and $\Delta\epsilon$ are of first order, the expressions $\Delta B^T \Delta B$ and $\Delta B^T \Delta\epsilon$ are of second order and one can assert that

$$
(48)\quad \begin{aligned} B^T B &\gg \Delta B^T \Delta B, \\ B^T \epsilon &\gg \Delta B^T \Delta \epsilon \end{aligned}
$$

This implies that as long as ρ is not too close to unity, the normal equations will be dominated by the leading terms and the solution vector δ will be only slightly dependent on the value of ρ. Hence in some situations the solution is but weakly affected by the presence of even rather moderate serial correlation. Be this as it may, the covariance matrix of the solution vector is very strongly dependent on the degree of serial correlation. Even when $B^T B$ dominates the normal equations, the covariance matrix of the parametric vector is given by

$$
(49)\quad \Sigma \simeq \frac{\sigma^2}{(1-\rho)^2} (B^T B)^{-1} .
$$

Thus, if ρ were actually equal to 0.9 and one were to ignore this fact in the adjustment, the solution vector itself would probably not be very much affected (for the factor $(1-\rho)^2$ of the leading terms on both sides of the normal equations would cancel out), but the covariance matrix associated with the solution would be incorrect by a factor of 100 (or $1/(1-\rho)^2$).

Because of the relative insensitivity of the solution vector to serial correlation, the residuals from the adjustment are also relatively insensitive to serial correlation (remember we are considering here an adjustment restricted to a single observational vector). This means that it is a sound and defensible practice to estimate the autoregressive function from residuals of a preliminary adjustment in which the observational errors are initially considered to be uncorrelated. The autoregressive function thus initially determined can then be fedback into the solution according to the development of this section and the resulting, more nearly correct normal equations can be solved to generate fresh residuals for a more refined autoregressive analysis. Clearly, this process can be iterated to stability (ordinarily, a single iteration is sufficient). Thus autoregressive feedback is an adaptive process that ultimately becomes independent of initial or a priori estimates of noise variance.

Returning to the normal equations (43), (45), we note that the extra computations entailed by rigorous consideration of a first order autoregressive process are surprisingly minor. The $B^T B$ and $B^T \epsilon$ terms would have to be computed in any case and the $\Delta B^T \Delta B$ and $\Delta B^T \Delta \epsilon$ terms, being of similar form, follow the same logic. Moreover, as is clear from (42), the equations can be formed in a cumulative manner, the $i+1$ st step of which would involve only observational equations from the i th and $i+1$ st observations. Hence, the formation of the normal equations for a first order process entails a scheme in which a moving pair of successive observational equations are processed at each step; similarily, a second order process entails a scheme in which a moving triplet of successive observational equations are processed at each step, and so on. A major benefit to be derived from the admission of an autoregressive process into the adjustment is the attainment of more realistic results from error propagation. In particular, it would permit one to extract whatever advantages are to be gained from moderately high sampling rates without paying the usual penalty of an absurdly optimistic estimation of output accuracies.

5.6 Normal Equations (Multi-Channel Case)

In the preceding section our consideration was limited to an adjustment involving only a single observational channel. Let us now turn to a multi-channel adjustment in which each channel is serially correlated. We assume, however, that serial crosscorrelation either does not exist, or else is ignored. Thus the various observational vectors are statistically independent of one another. Let us now consider the problem to be one of orbit determination in which a set of error coefficients is to be recovered for each observational channel. Let $\dot{\delta}$ denote the vector of orbital parameters and let $\ddot{\delta}_i$ denote the vector of the error parameters for the ith channel. Let us assume momentarily that only the ith channel is to be exercised in the adjustment. Then the normal equations can be written in the following partitioned form

$$(50)\quad \begin{bmatrix} \dot{N}_i & \bar{N}_i \\ \bar{N}_i^T & \ddot{N}_i \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \ddot{\delta}_i \end{bmatrix} = \begin{bmatrix} \dot{c} \\ \ddot{c}_i \end{bmatrix} .$$

Since this system corresponds to but a single observational channel we may apply the results of the preceeding section and assume that autoregressive feedback has been exercised and that the normal equations (50) re[illegible] the end product of the process. In this regard we would caution that in practice one would have to exercise temporary, coarse a priori constraints on the parametric vector in the process of single-channel autoregressive feedback because of the near singularity characteristic of single-channel orbital determination (especially when error models coefficients are also to be recovered). Such constraints would not be carried through the final set of normal equations (50) inasmuch as this system is not to be solved as an independent system.

Let us now assume that similar, individual and independent adjustments exercising autoregressive feedback were performed on data channels j and k which are also exercised on the same orbit. The corresponding normal equations may thus be written.

$$(51)\quad \begin{bmatrix} \dot{N}_j & \bar{N}_j \\ \bar{N}_j^T & \ddot{N}_j \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \ddot{\delta}_j \end{bmatrix} = \begin{bmatrix} \dot{c} \\ \ddot{c}_j \end{bmatrix} , \qquad \begin{bmatrix} \dot{N}_k & \bar{N}_k \\ \bar{N}_k^T & \ddot{N}_k \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \ddot{\delta}_k \end{bmatrix} = \begin{bmatrix} \dot{c} \\ \ddot{c}_k \end{bmatrix}$$

We observe that although equations (50) and (51) involve different parametric vectors, the subvector $\dot{\delta}$ is common to all three sets. Let us now define a composite parametric vector for all three channels as

$$(52)\quad \delta = \begin{bmatrix} \dot{\delta} \\ \ddot{\delta}_i \\ \ddot{\delta}_j \\ \ddot{\delta}_k \end{bmatrix} .$$

We may then augment each individual set of normal equations with appropriate sets of zero elements so that all will operate on the composite parametric vector. Thus we get

$$(53)\quad \begin{bmatrix} \dot{N}_i & \bar{N}_i & 0 & 0 \\ \bar{N}_i^T & \ddot{N}_i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \ddot{\delta}_i \\ \ddot{\delta}_j \\ \ddot{\delta}_k \end{bmatrix} = \begin{bmatrix} \dot{c} \\ \ddot{c}_i \\ 0 \\ 0 \end{bmatrix}$$

$$(54)\quad \begin{bmatrix} \dot{N}_j & 0 & \bar{N}_j & 0 \\ 0 & 0 & 0 & 0 \\ \bar{N}_j^T & 0 & \ddot{N}_j & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \ddot{\delta}_i \\ \ddot{\delta}_j \\ \ddot{\delta}_k \end{bmatrix} = \begin{bmatrix} \dot{c} \\ 0 \\ \ddot{c}_j \\ 0 \end{bmatrix}$$

$$(55)\quad \begin{bmatrix} \dot{N}_k & 0 & 0 & \bar{N}_k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{N}_k & 0 & 0 & \ddot{N}_k \end{bmatrix} \begin{bmatrix} \dot{\delta} \\ \ddot{\delta}_i \\ \ddot{\delta}_j \\ \ddot{\delta}_k \end{bmatrix} = \begin{bmatrix} \dot{c} \\ 0 \\ 0 \\ \ddot{c}_k \end{bmatrix}$$

We are now in a position to apply the rule that sets of normal equations formed from independent observational vectors and involving a common parametric vector can be summed to produce the normal equations appropriate to the simultaneous adjustment of the merged set of observations. Accordingly adding (53), (54), (55), we get

$$
(56) \qquad \left[\begin{array}{c|ccc}
\dot{N}_i + \dot{N}_j + \dot{N}_k & \bar{N}_i & \bar{N}_j & \bar{N}_k \\
\hline
\bar{N}_i^T & \ddot{N}_i & 0 & 0 \\
\bar{N}_j^T & 0 & \ddot{N}_j & 0 \\
\bar{N}_k^T & 0 & 0 & \ddot{N}_k
\end{array}\right]
\left[\begin{array}{c}
\dot{\delta} \\
\hline
\ddot{\delta}_i \\
\ddot{\delta}_j \\
\ddot{\delta}_k
\end{array}\right]
=
\left[\begin{array}{c}
\dot{c}_i + \dot{c}_j + \dot{c}_k \\
\hline
\ddot{c}_i \\
\ddot{c}_j \\
\ddot{c}_k
\end{array}\right]
$$

Except for the fact that we have yet to introduce the a priori constraint matrices $(\dot{W}, \ddot{W}_i, \ddot{W}_j, \ddot{W}_k)$ and supplementary discrepancy vectors $(\dot{\epsilon}, \ddot{\epsilon}_i, \ddot{\epsilon}_j, \ddot{\epsilon}_k)$; we recognize (56) as being of the form of a first order patterned system. By following a similar development we could show that a second order patterned system of normal equations will remain unaltered in form if autoregressive feedback is exercised for each of the observational channels. It follows, then, that the exercise of autoregressive feedback in SAGA does not seriously interrupt the general character of the data flow. In the computational outline provided in Section 6, we anticipate autoregressive feedback by the artifice of proceeding as if each channel were governed by a first order process having a prespecified coefficient of correlation. In general practice, each such coefficient would, for want of a better value, be set equal to zero in the initial reduction. However, if the autoregressive feedback option is exercised, the prespecified values would automatically be replaced in a repetition of the reduction by values estimated from the residuals computed from the initial reduction (iterated to convergence).

Although SAGA specifically considers only a first order autoregressive process, we would point out that in an article awaiting publication, J. Lynn of DBA shows that p iterations of a first order autoregressive process is equivalent to a pth order autoregressive process. Thus if first order autoregressive feedback should fail to produce uncorrelated secondary residuals, one could apply the process a second time to obtain the effect of second order autoregressive feedback, and so on until serially independent residuals are obtained.

As a final comment we would point out that autoregressive feedback is indifferent to the origin of the serial correlation in the observational channels. All or part of the correlation accounted for by the autoregressive process could consist of correlation induced by unmodelled systematic errors. Indeed, autoregressive feedback could in principle substitute completely for error modelling. This, however, is emphatically not to be recommended, for deletion of error coefficients from the model would induce such a high and persistent degree of serial correlation that the accuracy of the adjustment would be severely diluted. Hence, autoregressive feedback is more properly regarded as a powerful tool for accounting for the combined effects of natural serial correlation in the observations and induced serial correlation resulting from unavoidable deficiencies (hopefully small) of the mathematical model.

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6.0 COMPUTATIONAL OUTLINE OF SAGA

6.1 Introduction

The analytical basis for SAGA has been developed in the earlier sections. However, in these sections actual computational steps and flow are largely submerged in derivational detail. The purpose of this section is to extract from the derivation the essentials of the actual reduction. Our concern here is with the heart of the reduction and not with peripheral operations. We assume all appropriate preprocessing has been performed for each observational channel. Computational details of various preprocessing routines available to SAGA are provided in Part II of this report. Part II also provides details of the program itself — set up procedures, running instructions, flow charts, etc. In addition, it outlines results produced by SAGA for a tracking network of twenty stations participating in the observation of twenty seven passes of GEOS A. The network exercised the Goddard laser, the Geodetic Secor system and various MOTS and PC-1000 cameras (Geoceiver observations are not expected to become generally available until 1970).

No attempt has been made in the outline below to present the reduction in minute detail. The outline, rather, is intended to serve as a guide to a programmer who in turn is guided by an analyst having a sound understanding of the mathematical derivation of the reduction.

One point regarding the outline requires special clarification. This concerns the option for autoregressive feedback. In formulating the computations we have found it is convenient to proceed as if the autoregressive coefficients were known at the outset. In practice this will not be the case and values of zero will be initially employed for the coefficients. However, if the autoregressive feedback option is exercised, nonzero values of the coefficients will become available from the residuals resulting from the initial solution. Thus our formulation is designed to anticipate the possibility of revised autoregressive coefficients.

6.2 Computational Steps

A. Constants

a) All station data, baseline constraints, etc. are available from Master Survey File.

b) Standard schedules of sigmas and correlations of observations.

1a) σ_x, ρ_x standard schedule number 0 for x

1b) σ_x, ρ_x standard schedule number 1 for x

1c) σ_x, ρ_x standard schedule number 2 for x

2a) σ_y, ρ_y standard schedule number 0 for y

2b) σ_y, ρ_y standard schedule number 1 for y

2c) σ_y, ρ_y standard schedule number 2 for y

3a) σ_T standard schedule number 0 for optical timing

3b) σ_T standard schedule number 1 for optical timing

3c) σ_T standard schedule number 2 for optical timing

4a) $\sigma_r, \rho_r, \sigma_T$ standard schedule number 0 for ranges

4b) $\sigma_r, \rho_r, \sigma_T$ standard schedule number 1 for ranges

4c) $\sigma_r, \rho_r, \sigma_T$ standard schedule number 2 for ranges

c) Standard schedules of sigmas of error coefficients.

$$\left.\begin{array}{lll} \text{1a)} & \sigma_\alpha, \sigma_\omega, \sigma_\chi, \sigma_c, \sigma_t & \text{standard schedule number 0} \\ \text{1b)} & \sigma_\alpha, \sigma_\omega, \sigma_\chi, \sigma_c, \sigma_t & \text{standard schedule number 1} \\ \text{1c)} & \sigma_\alpha, \sigma_\omega, \sigma_\chi, \sigma_c, \sigma_t & \text{standard schedule number 2} \end{array}\right\} \text{optical}$$

$$\left.\begin{array}{lll} \text{2a)} & \sigma_{a_0}, \sigma_{a_1}, \sigma_{a_2}, \sigma_{a_3}, \sigma_{a_4}, \sigma_{a_5} & \text{standard schedule number 0} \\ \text{2b)} & \sigma_{a_0}, \sigma_{a_1}, \sigma_{a_2}, \sigma_{a_3}, \sigma_{a_4}, \sigma_{a_5} & \text{standard schedule number 1} \\ \text{2c)} & \sigma_{a_0}, \sigma_{a_1}, \sigma_{a_2}, \sigma_{a_3}, \sigma_{a_4}, \sigma_{a_5} & \text{standard schedule number 2} \end{array}\right\} \text{electronic}$$

d) Standard schedules of epochal sigmas.

1a) $\sigma_x, \sigma_y, \sigma_z, \sigma_{\dot{x}}, \sigma_{\dot{y}}, \sigma_{\dot{z}}$ standard schedule number 0

1b) $\sigma_x, \sigma_y, \sigma_z, \sigma_{\dot{x}}, \sigma_{\dot{y}}, \sigma_{\dot{z}}$ standard schedule number 1

1c) $\sigma_x, \sigma_y, \sigma_z, \sigma_{\dot{x}}, \sigma_{\dot{y}}, \sigma_{\dot{z}}$ standard schedule number 2

e) C_{mn}, S_{mn} $\left.\begin{array}{l} m = 0,1,2,3,4 \\ n = 0,1,2,3,4 \end{array}\right\}$ harmonic coefficients of gravity field

f) $\dot{\psi}$ = mean rotational rate of earth

B. <u>Data</u>

a) k = pass number

T_k = epoch of Kth pass

b) $X_k^0, Y_k^0, Z_k^0, \dot{X}_k^0, \dot{Y}_k^0, \dot{Z}_k^0$ = approximate values of inertial initial conditions at $t = T_k$

c) ξ_k = -1, 0, 1 or 2

= -1 if nonstandard schedule of epochal sigmas to be used

= 0, 1 or 2 if standard schedules 0, 1, 2 are to be used

d) If $\xi_k = -1$ read in alternative $\sigma_x, \sigma_y, \sigma_z, \sigma_{\dot{x}}, \sigma_{\dot{y}}, \sigma_{\dot{z}}$ for kth pass

e) $i = i_1, i_2, i_3, \ldots, i_{m_k}$ ($m_k \leq 15$) schedule of stations participating on kth pass

f) Start and stop times of tracking intervals on kth pass from ith station (up to 4 intervals allowed).

t_{ap}, t_{bp} = first and last times of pth interval (maximum p = 4)

g) $q = 0$ indicates optical data

$q = 1$ indicates electronic data

<u>If q = 0</u>, data block for kth pass from ith station consists of the following.

i) t_j, α_j, δ_j (time, right ascension, declination) $j = 1, 2, \ldots, m_i$

ii) α_p = $-1, 0, 1$ or 2 (note $p=1,2,3$ or 4)

= -1, if nonstandard schedule of observational sigmas is to be used for pth interval

= 0, 1 or 2 according to the standard schedule (0, 1, 2) to be used for pth interval

iii) If $\alpha_p = 1$, use alternative schedule σ_x, ρ_x, for pth interval

σ_y, ρ_y

iv) β_p = $-1, 0, 1$ or 2

= -1, if nonstandard schedule of error model sigmas is to be used for pth interval

= 0, 1 or 2 according to the standard schedule (0, 1, 2) to be used for pth interval

v) β_p = -1, use nonstandard schedule σ_α, σ_ω, σ_χ, σ_c, σ_t

vi) γ_p = 0 if autoregressive feedback option is not to be exercised on pth interval

= 1 otherwise

vii) c_i = nominal focal length of camera

If q = 1, data block for kth pass from ith station consists of the following.

i) t_j, r_j (time, range) $j = 1, 2, \ldots, m_i$

ii) $\alpha_p, \beta_p, \gamma_p$ are defined as in optical case except that they refer now to ranging observations.

C. Computations

Part I. Orbital Integration

Read in data for kth pass. Perform orbital integration with initial conditions given by (b). Results of integration are:

$$(1) \qquad \underset{(3,\,p+1)}{P} = \begin{Bmatrix} a_0 & a_1 & a_2 & \ldots & a_p \\ b_0 & b_1 & b_2 & \ldots & b_p \\ c_0 & c_1 & c_2 & \ldots & c_p \end{Bmatrix} = \text{coefficients of } X, Y, Z \text{ polynomials (degree p)}$$

$$(2) \quad \underset{(s,s(q+1))}{\Omega} = \begin{Bmatrix} \Omega_{11} & \Omega_{12} \cdots \Omega_{1s} \\ \Omega_{21} & \Omega_{22} \cdots \Omega_{2s} \\ \Omega_{s1} & \Omega_{s2} \cdots \Omega_{ss} \end{Bmatrix} = \text{coefficients of matrizant polynomials}$$

in which

$$(3) \quad \underset{(1,q+1)}{\Omega_{\ell_m}} = \{\alpha_0 \; \alpha_1 \; \cdots \; \alpha_q\} = \text{row vector of coefficients}$$

the matrix $\underset{(s,s(q+1))}{\Omega}$ can also be partitioned as

$$(4) \quad \Omega = \big(\underset{(s,sq+s)}{\Omega_0} \quad \underset{(s,sq+s)}{\Omega_1} \quad \underset{(s,sq+s)}{\Omega_2} \big) .$$

The matrices P and Ω are stored for later use.

Part IIa. Optical Reduction (use if q = 0; if q = 1; go on to Part IIa)

(5) Use dummy camera approach to project each of up to four pic as onto hypothetical plates. Aim camera axis at nominal midpoint to get α, ω. Let x be initially zero, but then modify it so that final x axis is aligned with trace (as in continuous traces).

Results of (5) are sets of plate coordinates

(6) $t_j, x_j, y_j \quad j = 1, 2, \ldots, m_\ell$

and orientation matrices

$$\begin{Bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{Bmatrix}_\ell \quad \ell = 1, 2, \ldots, p \quad (\text{max } p = 4)$$

(p = number plates on pass) .

For pth tracking interval (plate) compute ephemeris for each point within interval. Thus for time t_j compute:

$$(7)\quad \begin{bmatrix} x & \dot{x} \\ y & \dot{y} \\ z & \dot{z} \end{bmatrix}_j = \begin{bmatrix} a_0 & a_1 & \cdots & a_p \\ b_0 & b_1 & \cdots & b_p \\ c_0 & c_1 & \cdots & c_p \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \tau & 1 \\ \tau^2 & 2\tau \\ \vdots & \\ \tau^p & p\tau^{p-1} \end{bmatrix}_j$$

in which

$$\tau_j = t_j - T_k \quad (T_k = \text{epoch for } k\text{th pass}).$$

Set up matrices

$$(8)\quad R_j = \begin{Bmatrix} \cos\psi\tau_j & \sin\psi\tau_j & 0 \\ -\sin\psi\tau_j & \cos\psi\tau_j & 0 \\ 0 & 0 & 1 \end{Bmatrix}, \qquad \dot{R}_j = \psi \begin{Bmatrix} -\sin\psi\tau_j & \cos\psi\tau_j & 0 \\ -\cos\psi\tau_j & -\sin\psi\tau_j & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$

ψ = rotational rate of earth.

Transform inertial coordinates to earth fixed coordinates

$$(9)\quad \begin{bmatrix} X \\ Y \\ Z \\ \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix}_j = \begin{bmatrix} R_j & 0 \\ \dot{R}_j & R_j \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_j .$$

Evaluate the matrizant for time τ_j

$$(10)\quad \underset{(3,9)}{\Omega_j} = \begin{bmatrix} \Omega_{11j} & \Omega_{12j} & \cdots & \Omega_{19j} \\ \Omega_{21j} & \Omega_{22j} & \cdots & \Omega_{29j} \\ \Omega_{31j} & \Omega_{32j} & \cdots & \Omega_{39j} \end{bmatrix}$$

in which

$$(11)\quad \underset{(1,1)}{\Omega_{\ell_m j}} = \underset{(1,q+1)}{\Omega_{\ell_m}} \begin{bmatrix} 1 \\ \tau_j \\ \tau_j^2 \\ \vdots \\ \tau_j^q \end{bmatrix} .$$

Transform the matrizant Ω_j to earth fixed coordinates.

$$(12)\quad \underset{(3,9)}{\Phi_j} = \underset{(3,3)}{R_j}\ \underset{(3,9)}{\Omega_j} .$$

From the X_j, Y_j, Z_j obtained from (9) and the X_i^c, Y_i^c, Z_i^c obtained from the master survey file, compute for pth plate

$$(13)\quad \begin{Bmatrix} m \\ n \\ q \end{Bmatrix}_{pj} = \begin{Bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{Bmatrix}_p \begin{Bmatrix} X_j - X_i^c \\ Y_j - Y_i^c \\ Z_j - Z_i^c \end{Bmatrix}$$

and from these the plate coordinates

$$(14)\quad \begin{pmatrix} x^{00} \\ y^{00} \end{pmatrix}_{pj} = \frac{c}{q_{pj}} \begin{bmatrix} m \\ n \end{bmatrix}_{pj} \quad (c = \text{focal length of camera}) .$$

Set up matrix

$$(15)\quad \begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \end{bmatrix}_{pj} = \frac{c}{q_{pj}} \begin{bmatrix} 1 & 0 & -x_j^{00}/c \\ 0 & 1 & -y_j^{00}/c \end{bmatrix} \begin{bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{bmatrix}_p$$

$$(16)\quad \begin{bmatrix} \dot{x}_{pj} \\ \dot{y}_{pj} \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \end{bmatrix}_p \begin{pmatrix} \dot{X}_j \\ \dot{Y}_j \\ \dot{Z}_j \end{pmatrix} .$$

Using designated schedule of sigmas, compute:

$$
(17)\qquad
\begin{aligned}
\sigma_{x_{pj}} &= (\sigma_x^2 + \dot{x}_j^2\,\sigma_T^2)^{\frac{1}{2}}\\
\sigma_{y_{pj}} &= (\sigma_y^2 + \dot{y}_j^2\,\sigma_T^2)^{\frac{1}{2}}\,.
\end{aligned}
$$

Set up the following coefficient matrices:

$$
(18)\qquad \underset{(1,3)}{B^{(1)}_{pj}(x)} = \frac{-1}{\sigma_{x_{pj}}}\,(f_1\;f_2\;f_3)_{pj}\,.
$$

$$
(19)\qquad \underset{(1,9)}{B^{(2)}_{pj}(x)} = \underset{(1,3)}{-B^{(1)}_{pj}(x)}\;\underset{(3,9)}{\Phi_j} = \left\{\underset{(1\times 3)}{B^{(2a)}_{pj}(x)} \;\vdots\; \underset{(1\times 6)+1}{B^{(2b)}_{pj}(x)}\right\}
$$

$$
(20)\qquad \underset{(1,1)}{B^{(3)}_{pj}(x)} = \frac{1}{\sigma_{x_{pj}}}\left\{\underset{(1,1)}{\dot{x}_{pj}}\right\}
$$

$$
(21)\qquad \underset{(1,4)}{B^{(4)}_{pj}(x)} = \frac{1}{\sigma_{x_{pj}}}\left\{-\frac{(c^2+x^2)}{c}\quad \frac{xy}{c}\quad +y\quad \frac{x}{c}\right\}_{pj}
$$

$$
(22)\qquad \epsilon_{pj}(x) = \frac{1}{\sigma_{x_{pj}}}\left\{x^{oc}_{pj} - x_{pj}\right\}.
$$

In terms of the above set up the composite matrix

$$
(23)\qquad \underset{(1,13+4p)}{B_{pj}(x)} = \left\{\underset{(1,3)}{B^{(1)}_{pj}(x)}\;\; \underset{(1,3)}{B^{(2a)}_{pj}(x)} \;\vdots\; \underbrace{\underset{(1,6)}{B^{(2b)}_{pj}(x)} \;\vdots\; \underset{(1,1)}{B^{(3)}_{pj}(x)}\;\; \underset{(1,4)}{\xi_{1p}B^{(4)}_{pj}(x)}\;\; \underset{(1,4)}{\xi_{2p}B^{(4)}_{pj}(x)}\;\; \underset{(1,4)}{\xi_{3p}B^{(4)}_{pj}(x)}\;\; \underset{(1,4)}{\xi_{4p}B^{(4)}_{pj}(x)}}_{\ddot{B}_{pj}(x)}\right\}
$$

in which

$$\xi_{ab} = 1 \text{ if } a=b$$
$$= 0 \text{ if } a \neq b.$$

Perform the above computations for point $j+1$ also, getting:

(24) $B_{p,j+1}(x)$ and $\epsilon_{p,j+1}(x)$.

Compute:

(25) $\Delta B_{pj}(x) = B_{p,j+1}(x) - B_{p,j}(x)$

(26) $\Delta \epsilon_{pj}(x) = \epsilon_{p,j+1}(x) - \epsilon_{p,j}(x)$.

In terms of the above compute:

(27) $N_{pj}(x) = (1-\rho_{px})^2 B^T_{pj}(x) B_{pj}(x) + \rho_{px} \Delta B^T_{pj}(x) \Delta B_{pj}(x)$

in which ρ_{px} denotes correlation coefficient ρ_x from input schedule. Superscript p is added to indicate that in later solution ρ_x may differ from plate to plate. Similarly compute:

(28) $c_{pj}(x) = (1-\rho_{px})^2 B^T_{pj}(x) \epsilon_{pj}(x) + \rho_{px} \Delta B^T_{pj}(x) \Delta \epsilon_{pj}(x)$.

As $N_{pj}(x)$ and $c_{pj}(x)$ are generated, evaluate the sums

(29) $$N_p(x) = \left\{ \sum_{j=a_p}^{b_p} N_{pj}(x) \right\} + \rho_{px}(1-\rho_{px})(B^T_{ap}(x) B_{ap}(x) + B^T_{bp}(x) B^T_{bp}(x))$$

(30) $$c_p(x) = \left\{ \sum_{j=a_p}^{b_p} \epsilon_{pj}(x) \right\} + \rho_{px}(1-\rho_{px})(B^T_{ap}(x) \epsilon_{ap}(x) + B^T_{bp}(x) \epsilon_{bp}(x))$$

In which $\left\{ \begin{array}{l} B_{ap}(x) \text{ and } B_{bp}(x) \\ \epsilon_{ap}(x) \text{ and } \epsilon_{bp}(x) \end{array} \right\}$ correspond to first and last points on pth plate.

Computations precisely paralleling those (18) thru (30) are performed for the y coordinate. Thus:

$$(18a)\quad \underset{(1,3)}{B^{(1)}_{p\,j}(y)} = \frac{1}{\sigma_{y_{p\,j}}} \left(f_1' \;\; f_2' \;\; f_3'\right)_{p\,j}$$

$$(19a)\quad \underset{(1,9)}{B^{(2)}_{p\,j}(y)} = -\underset{(1,3)}{B^{(1)}_{p\,j}(y)} \;\; \underset{(1,9)}{\Phi_j} = \left\{ \underset{(1,3)}{B^{(2a)}_{p\,j}(y)} \;\Big\vdots\; \underset{(1,6)}{B^{(2b)}_{p\,j}(y)} \right\}$$

$$(20a)\quad \underset{(1,1)}{B^{(3)}_{p\,j}(y)} = \frac{1}{\sigma_{y_{p\,j}}} \left\{ \dot{y}_{p\,j} \right\}$$

$$(21a)\quad B^{(4)}_{p\,j}(y) = \frac{1}{\sigma_{y_{p\,j}}} \left\{ \frac{xy}{c} \;\; \frac{(c^2+y^2)}{c} \;\; x \;\; \frac{y}{c} \right\}_{p\,j}$$

$$(22a)\quad \epsilon_{p\,j}(y) = \frac{1}{\sigma_{y_{p\,j}}} \left\{ y^{00}_{p\,j} - y_{p\,j} \right\}.$$

In terms of the above set up

$$\underset{(1,13+4p)}{B_{p\,j}(y)} = \text{same as (23) with } y \text{ replacing } x .$$

Equations (24a) — (30a) are the same as (24) — (30) with y replacing x. Ultimate result is $N_p(y)$, $c_p(y)$. Combine these with $N_p(x)$ and $\epsilon_p(x)$ to get

$$(31)\quad \begin{aligned} \underset{(13+4p)(13+4p)}{N_p} &= N_p(x) + N_p(y) \\ \underset{(13+4p,\,1)}{c_p} &= c_p(x) + c_p(y). \end{aligned}$$

This completes initial computations for pth plate from ith station for kth pass. Perform above for all plates covering pass (maximum p=4) from ith station. Add together all N_p and c_p's. End result is N_{ik}, c_{ik} in which subscripts i (station number) and k (pass number) have now been introduced. Proceed to Part III.

Part IIb. Electronic Reduction (if q=1, the following reduction is performed)

For time t_j repeat steps (7) through (12) of Part II. Then proceed as follows. From: X_j, Y_j, Z_j computed from (9) evaluate:

(32) $r_j^{oo} = [(X_j - X_i^c)^2 + (Y_j - Y_i^c)^2 + (Z_j - Z_i^c)^2]^{\frac{1}{2}}$

(33) $\lambda_j = (X_j - X_i^c)/r_j^{oo}, \quad \mu_j = (Y_j - Y_i^c)/r_j^{oo}, \quad \nu_j = (Z_j - Z_i^c)/r_j^{oo}$

(34) $\dot{r}_j^{oo} = \lambda_j \dot{X}_j + \mu_j \dot{Y}_j + \nu_j \dot{Z}_j$

(35) $\Delta Z_i^c = \dfrac{a\, e^2 \sin \Phi_i}{\sin^2 \Phi_i}$ $\quad \begin{pmatrix} a, e^2 = \text{parameters of spheroid} \\ \Phi_i = \text{geographic latitude} \end{pmatrix}$

(36) $r_i^c = [(X_i^c)^2 + (Y_i^c)^2 + (Z_i^c + \Delta Z_i^c)^2]^{\frac{1}{2}}$

(37) $\lambda_i^c = X_i^c / r_i^c, \quad \mu_i^c = Y_i^c / r_i^c, \quad \nu_i^c = (Z_i^c + \Delta Z_i^c)/r_i^c$, direction cosines to station,

(38) $\sin E_j = \lambda_i^c \lambda_j + \mu_i^c \mu_j + \nu_i^c \nu_j$.

If ranges are not corrected for tropospheric refraction, evaluate:

(39) $N_0 - 1 = 10^{-6} \left[103.5 \dfrac{(P_0 - \epsilon_0)}{T_0} + 86.3 \left(1 + \dfrac{5748}{T_0}\right) \dfrac{\epsilon_0}{T_0} \right]$

where

P_o = pressure (mm) at station

T_o = temperature (K) at station

ϵ_o = water vapor pressure (mm) at station.

For optical (laser) ranges set $\epsilon_o = 0$ in (39) and replace second term by $0.58/\lambda^2$ where λ is the wavelength of the light in microns.

(40) $H_o = 29.2\,(T_o - 30)$

(41) $\alpha = 2(N_o - 1)\,H_o$

$K^2 = 4(H_o/\dot{r}_j)$

(42) $$f(E_j) = \frac{1}{\sin E_j + [\sin^2 E_j + K^2]^{\frac{1}{2}}}$$

(43) $\Delta r_j = \alpha f(E_j)$ = refraction correction (to be used in equation (48)).

If ranges have been corrected for tropospheric refraction, set $\alpha = 0$ in (43). For pth tracking interval set up the matrices

(44) $$B^{(1)}_{pj} = \frac{1}{\sigma_{r_{pj}}} [\lambda_j \;\; \mu_j \;\; \nu_j]$$

in which σ_{r_p} denotes the expression $(\sigma_r^2 + \dot{r}^2 \sigma_T^2)^{\frac{1}{2}}$. Set up

(45) $$B^{(2)}_{pj} = -B_{pj}\,\Phi_j = \left\{B^{(2a)}_{pj} \;\; B^{(2b)}_{pj}\right\}$$

(46) $$B^{(3)}_{pj} = \frac{1}{\sigma_{r_{pj}}} \{\tau \;\; \tau^2 \;\; r^{00} \;\; \dot{r}^{00} \;\; f(E)\}_{pj} \qquad (\tau_{pj} = t_j - T_k)$$

(47) $$B^{(4)}_{pj} = \frac{1}{\sigma_{r_{pj}}} \{1\}$$

(48) $$\epsilon_{pj} = \frac{1}{\sigma_{r_{pj}}} \left\{r^{00}_{pj} - (r_j + \Delta r_j)\right\},$$ where r_j is the measured range.

In terms of the above set up the composite matrix:

$$(49)\quad \underset{(1,17+p)}{B^{(1)}_{pj}} = \Big\{ \underset{(1,3)}{B^{(1)}_{pj}}\ \underset{(1,3)}{B^{(2a)}_{pj}}\ \underset{(1,3)}{B^{(2b)}_{pj}}\ \underset{(1,5)}{B^{(3)}_{pj}}\ \underbrace{\underset{(1,1)}{\xi_{1p} B^{(4)}_{pj}}\ \underset{(1,1)}{\xi_{2p} B^{(4)}_{pj}}\ \underset{(1,1)}{\xi_{3p} B^{(4)}_{pj}}\ \underset{(1,1)}{\xi_{4p} B^{(4)}_{pj}}}_{\ddot{B}_j} \Big\}$$

in which

$$\xi_{ab} = 1 \text{ if } a=b$$
$$= 0 \text{ if } a \neq b .$$

Perform the above computations for point j+1 also, getting:

$$(50)\quad B_{p,\,j+1},\ \epsilon_{p,\,j+1} .$$

Next computations parallel those indicated in (25) -- (30). The end results are:

$$(51)\quad N_p = \sum_{j=a_p}^{b_p} N_{pj} + \rho_{pr}\,(1-\rho_{pr})\,[B^T_{a,p}\, B_{a,p} + B^T_{b,p}\, B_{b,p}]$$

$$(52)\quad c_p = \sum_{j=a_p}^{b_p} c_{pj} + \rho_{pr}\,(1-\rho_{pr})\,[B^T_{a,p}\, \epsilon_{a,p} + B^T_{b,p}\, \epsilon_{b,p}]$$

in which a, p and b, p denote first and last points of p th tracking interval. This completes initial computations for p th tracking interval from i th station. Add together all N_p's and c_p's. End result is N_{ik}, c_{ik} in which subscripts i (station number) and k (pass number) have now been introduced. Proceed to Part III.

Part III. Second Order Partitioned Regression

From Part IIa or Part IIb the matrices N_{ik}, c_{ik} are generated from the observations of the k th pass from the i th station. These matrices can be partitioned as follows:

$$
(53)\quad N_{ik} = \begin{bmatrix} \underset{(6,6)}{\hat{U}_{ik}} & \underset{(6,6)}{\dot{U}_{ik}} & \underset{(6,\ell)}{\bar{U}_{ik}} \\ \underset{(6,6)}{\dot{U}^T_{ik}} & \underset{(6,6)}{\dot{N}_{ik}} & \underset{(6,\ell)}{\bar{N}_{ik}} \\ \underset{(\ell,6)}{\bar{U}^T_{ik}} & \underset{(\ell,6)}{\bar{N}^T_{ik}} & \underset{(\ell,\ell)}{\ddot{N}_{ik} + \ddot{W}_{ik}} \end{bmatrix} \qquad c_{ik} = \begin{bmatrix} \underset{(1,6)}{\hat{c}_{ik}} \\ \underset{(1,6)}{\dot{c}_{ik}} \\ \underset{(1,\ell)}{\ddot{c}_{ik}} \end{bmatrix}
$$

in which ℓ depends on the number of tracking intervals and the type of data (optical or electronic). Also, the apriori weight matrix $\ddot{W}_{ik}$ has been introduced. This is computed from:

$$
(54)\quad \underset{\text{(for optical)}}{\ddot{W}_{ik}} = \text{diag}\left(\overbrace{\frac{1}{\sigma_t^2}\ \frac{1}{\sigma_\alpha^2}\ \frac{1}{\sigma_\omega^2}\ \frac{1}{\sigma_\chi^2}\ \frac{1}{\sigma_c^2}}^{\text{Plate 1}}\quad \overbrace{\frac{1}{\sigma_\alpha^2}\ \frac{1}{\sigma_\omega^2}\ \frac{1}{\sigma_\chi^2}\ \frac{1}{\sigma_c^2}}^{\text{Plate 2}} \cdots \overbrace{\frac{1}{\sigma_\alpha^2}\ \frac{1}{\sigma_\omega^2}\ \frac{1}{\sigma_\chi^2}\ \frac{1}{\sigma_c^2}}^{\text{Plate p (max p=4)}} \right)
$$

$$
(55)\quad \underset{\text{(for electronic)}}{\ddot{W}_{ik}} = \text{diag}\left(\frac{1}{\sigma_{a_1}^2}\ \frac{1}{\sigma_{a_2}^2}\ \frac{1}{\sigma_{a_3}^2}\ \frac{1}{\sigma_{a_4}^2}\ \frac{1}{\sigma_{a_5}^2}\ \overbrace{\frac{1}{\sigma_{a_0}^2}\ \frac{1}{\sigma_{a_0}^2} \cdots \frac{1}{\sigma_{a_0}^2}}^{\text{one for each unbroken tracking interval}} \right)
$$

$\ddot{\delta}^{(q)}_{ik}$ = corrections resulting from q th iterative cycle ($\ddot{\delta}^{(0)}_{ik} = 0$).

In terms of elements of N_{ik}, c_{ik} compute:

$$
(56)\quad \begin{bmatrix} \underset{(6,6)}{[\dot{N}]_{ik}} & \underset{(6,6)}{[\bar{N}]_{ik}} \\ \underset{(6,6)}{[\bar{N}]^T_{ik}} & \underset{(6,6)}{[\dot{N}]_{ik}} \end{bmatrix} = \begin{bmatrix} \hat{U}_{ik} & \dot{U}_{ik} \\ \dot{U}^T_{ik} & \dot{N}_{ik} \end{bmatrix} - \begin{bmatrix} \bar{U}_{ik} \\ \bar{N}_{ik} \end{bmatrix} (\ddot{N}_{ik} + \ddot{W}_{ik})^{-1} \left[\bar{U}^T_{ik} \quad \bar{N}^T_{ik} \right]
$$

$$
(57)\quad \begin{bmatrix} \underset{(1,3)}{[\dot{c}]_{ik}} \\ \\ \underset{(1,6)}{[\ddot{c}]_{ik}} \end{bmatrix} = \begin{bmatrix} \hat{c}_{ik} \\ \\ \dot{c}_{ik} \end{bmatrix} - \begin{bmatrix} \bar{U}_{ik} \\ \\ \bar{N}_{ik} \end{bmatrix} (\ddot{N}_{ik} + \ddot{W}_{ik})^{-1} \left[\ddot{c}_{ik} - \ddot{W}_{ik} \sum_{\ell=1}^{\ell-1} \ddot{\delta}_{ik}^{(\ell)} \right]
$$

This completes the separate computations for the observations of the kth pass from the ith station. Repartition the matrices generated in (56) and (57) as:

$$
(58)\quad \left[\begin{array}{c:c} \underset{(6,6)}{[\dot{N}]_{ik}} & \underset{(6,6)}{[\bar{N}]_{ik}} \\ \hdashline \underset{(6,6)}{[\bar{N}^T]_{ik}} & \underset{(6,6)}{[\ddot{N}]_{ik}} \end{array}\right] = \left[\begin{array}{cc:c} \underset{(3,3)}{[\dot{N}_{11}]_{ik}} & \underset{(3,3)}{[\dot{N}_{12}]_{ik}} & \underset{(3,6)}{[\bar{N}_1]_{ik}} \\ \underset{(3,3)}{[\dot{N}_{12}]^T_{ik}} & \underset{(3,3)}{[\dot{N}_{22}]_{ik}} & \underset{(3,6)}{[\bar{N}_2]_{ik}} \\ \hdashline \underset{(6,3)}{[\bar{N}_1]^T_{ik}} & \underset{(6,3)}{[\bar{N}_2]_{ik}} & \underset{(6,6)}{[\ddot{N}]_{ik}} \end{array}\right], \quad \left[\begin{array}{c} \underset{(6,1)}{[\dot{c}]_{ik}} \\ \hdashline \underset{(6,1)}{[\ddot{c}]_{ik}} \end{array}\right] = \left\{\begin{array}{c} \underset{(3,1)}{[\dot{c}_1]_{ik}} \\ \underset{(3,1)}{[\dot{c}_2]_{ik}} \\ \hdashline \underset{(6,1)}{[\ddot{c}]_{ik}} \end{array}\right\}.
$$

As the matrices defined in (58) are generated by each station participating in the observation of the kth pass, set up the matrices:

$$
(59)\quad \underset{(3m_k+3,\,3m_k+3)}{[\dot{N}]} = \begin{bmatrix} [\dot{N}_{11}]_{i_1 k} & 0 & \cdots & 0 & [\dot{N}_{12}]_{i_1 k} \\ 0 & [\dot{N}_{11}]_{i_2 k} & \cdots & 0 & [\dot{N}_{12}]_{i_2 k} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & [\dot{N}_{11}]_{i_{m_k} k} & [\dot{N}_{12}]_{i_{m_k} k} \\ [\dot{N}_{12}]^T_{i_1 k} & [\dot{N}_{12}]^T_{i_2 k} & \cdots & [\dot{N}_{12}]^T_{i_{m_k} k} & \sum_{j=1}^{m_k} [\dot{N}_{22}]_{i_j k} \end{bmatrix}
$$

$$\underset{(3m_k+3,\,6)}{[\bar{N}]_k} = \begin{bmatrix} [\bar{N}_1]_{i_1 k} \\ [\bar{N}_1]_{i_2 k} \\ \vdots \\ [\bar{N}_1]_{i_{m_k} k} \\ \sum_{j=1}^{m_k} [\bar{N}_2]_{i_j k} \end{bmatrix}$$

(60) $$\underset{(6,6)}{[\ddot{N}]_k} = \underset{(6,6)}{[\ddot{N}]_{i_1 k}} + \underset{(6,6)}{[\ddot{N}]_{i_2 k}} + \ldots + \underset{(6,6)}{[\ddot{N}]_{i_{m_k} k}}$$

(61) $$\underset{(3m_k+3,\,1)}{[\dot{c}]_k} = \begin{bmatrix} [\dot{c}_1]_{i_1 k} \\ [\dot{c}_1]_{i_2 k} \\ \vdots \\ [\dot{c}_1]_{i_{m_k} k} \\ \sum_{j=1}^{m_k} [\dot{c}_2]_{i_j k} \end{bmatrix}, \quad \underset{(6,1)}{[\ddot{c}]_k} = [\ddot{c}]_{i_1 k} + [\ddot{c}]_{i_2 k} + \ldots + [\ddot{c}]_{i_{m_k} k} .$$

Set up the a priori weight matrix of orbital parameters using schedule indicated in input:

(62) $$\underset{(6,6)}{\ddot{W}_k} = \operatorname{diag}\left(\frac{1}{\sigma^2_{x_0}} \; \frac{1}{\sigma^2_{y_0}} \; \frac{1}{\sigma^2_{z_0}} \; \frac{1}{\sigma^2_{\dot{x}_0}} \; \frac{1}{\sigma^2_{\dot{y}_0}} \; \frac{1}{\sigma^2_{\dot{z}_0}}\right) .$$

Compute:

(63) $$\underset{(3m_k+3,\,3m_k+3)}{[S]_k} = [\dot{N}]_k - [\bar{N}]_k \left\{[\ddot{N}]_k + \ddot{W}_k\right\}^{-1} [\bar{N}]_k^T$$

$$(64)\quad \underset{(3n_k+3,1)}{[c_k]} = [\dot{c}]_k - [\bar{N}]_k \left\{[\ddot{N}]_k + \ddot{W}_k\right\}^{-1} \left\{[\ddot{c}]_k - \ddot{W}_k \sum_{\ell=1}^{\ell-1} \dot{\delta}_k^{(\ell)}\right\}.$$

This completes computations for k th pass. Merge $[S]_k$ in the master survey file (augmented by 3 rows ascolumns to accommodate the coordinates of the earth's center of mass.) Repeat the above process until all passes have been absorbed into master survey matrix. Final result is 3m+3, 3m+3 matrix $S+\dot{W}$ ($\dot{W}$ is augmented to account for center of mass) and 3m+3, 1 vector c. Compute vector of corrections to survey and center of mass (superscript (ℓ) is used to designate the ℓ th iteration of the solution).

$$(65)\quad \underset{(3m+3,3m+3)}{\hat{\delta}^{(\ell)}} = (S+\dot{W})^{-1} \left(c - \dot{W} \sum_{\ell=1}^{\ell-1} \hat{\delta}^{(\ell)}\right) \text{ where } \hat{\delta}^{(0)} = 0.$$

For each pass, in turn, compute the vector of corrections to the orbital parameters:

$$(66)\quad \underset{(6,1)}{\dot{\delta}_k^{(\ell)}} = \left\{[\ddot{N}]_k + \ddot{W}_k\right\}^{-1} \left\{[\ddot{c}]_k - \ddot{W}_k \sum_{\ell=1}^{\ell-1} \dot{\delta}_k^{(\ell)}\right\} - \left\{[\ddot{N}]_k + \ddot{W}_k\right\}^{-1} [\bar{N}]_k^T [\hat{\delta}]_k^{(\ell)}$$

in which $[\hat{\delta}]_k$ denotes that portion of $\hat{\delta}$ containing the set of stations participating on the k th pass, i.e.:

$$[\hat{\delta}]_k = \begin{Bmatrix} \hat{\delta}_{i_1} \\ \hat{\delta}_{i_2} \\ \vdots \\ \hat{\delta}_{i_{n_k}} \\ \hat{\delta}_{00} \end{Bmatrix}.$$

The error parameters for the i th station and k th pass are computed from:

$$(67)\qquad \underset{(\ell,1)}{\ddot{\delta}^{(\ell)}_{ik}} = \underset{(\ell,\ell)}{(\ddot{N}_{ik}+\ddot{W}_{ik})^{-1}} \left\{ \underset{(\ell,1)}{\ddot{c}_{ik}} - \ddot{W}_{ik} \sum_{\ell}^{\ell-1} \ddot{\delta}^{(\ell)}_{ik} \right\} - (\ddot{N}_{ik}+\ddot{W}_{ik})^{-1}\, \bar{N}^{T}_{ik}\, \dot{\delta}^{(\ell)}_{k}.$$

Add corrections provided by $\hat{\delta}^{(\ell)}$ to survey and corrections provided by $\ddot{\delta}^{(\ell)}_{k}$ to initial conditions of k th pass. From error parameters compute the following corrections for i th station and k th pass.

$$(68)\qquad \left.\begin{aligned} \delta_{x_{p_j}} &= \sigma_{x_{p_j}} \underset{(1,\ell)}{\ddot{B}_j(x)} \sum_{\ell=1}^{\ell-1} \underset{(\ell,1)}{\ddot{\delta}^{(\ell)}} \\ \delta_{y_{p_j}} &= \sigma_{y_{p_j}} \underset{(1,\ell)}{\ddot{B}_j(y)} \sum_{\ell=1}^{\ell-1} \underset{(\ell,1)}{\ddot{\delta}^{(\ell)}} \end{aligned}\right\} \text{ optical case}$$

$$(69)\qquad \delta_{r_{p_j}} = \sigma_{r_{p_j}} \ddot{B}_j \sum_{\ell=1}^{\ell-1} \ddot{\delta}^{(\ell)} \quad \Big| \text{ electronic case}$$

in which the $\ddot{B}_j$ matrices are based on the latest approximations for survey and orbit. The residuals for the k th tracking interval are then computed from:

$$(70)\qquad \left.\begin{aligned} v_{x_{p_j}} &= \frac{\sigma_x^2}{\sigma_x^2+\dot{x}_j^2\,\sigma_T^2} \left\{x^{oo}_{p_j} - (x_{p_j} + \delta x_{p_j})\right\} \\ v_{T_{p_j}} &= \frac{\dot{x}_j^2\,\sigma_T^2}{\sigma_x^2+\dot{x}_j^2\,\sigma_T^2} \left\{x^{oo}_{p_j} - (x_{p_j} + \delta x_{p_j})\right\} \\ v_{y_{p_j}} &= \left\{y^{oo}_{p_j} - (y_{p_j} + \delta y_{p_j})\right\} \end{aligned}\right\} \text{ optical case}$$

$$(71)\qquad \left.\begin{aligned} v_{r_{p_j}} &= \frac{\sigma_r^2}{\sigma_r^2+\dot{r}^2\,\sigma_T^2} \left\{r^{oo}_{p_j} - (r_{p_j} + \delta r_{p_j})\right\} \\ v_{T_{p_j}} &= \frac{\dot{r}^2\,\sigma_T^2}{\sigma_r^2+\dot{r}^2\,\sigma_T^2} \left\{r^{oo}_{p_j} - (r_{p_j} + \delta r_{p_j})\right\} \end{aligned}\right\} \text{ electronic case}$$

In (70), (71) the quantities $x^{00}_{p_j}$, $y^{00}_{p_j}$ and $r^{00}_{p_j}$ are based on the most recent values for survey and orbit.

Compute grand mean error for ℓ th iteration from:

$$(72) \quad \sigma^{(\ell)} = \left\{ \frac{\text{sum of squares of weighted residuals for all stations and all passes}}{\text{degrees of freedom}} \right\}^{\frac{1}{2}}$$

degrees of freedom = 2(total number optical points) + (total number of ranges)
-6(number of passes) - (total number error parameters)
-3(number stations).

Solution is considered to have converged when:

$$(73) \quad \left| \sigma^{(\ell-1)} - \sigma^{(\ell)} \right| \leqq \xi ,$$

a prespecified constant or when five iterations have been performed, in the event the inequality is not satisfied for $\ell \leqq 5$.

<u>Part IV. Autoregressive Feedback</u> (Option exercised only if $\gamma_p = 1$ for tracking interval)

After solution has converged, compute final primary residuals from (70) or (71). For each tracking interval compute:

$$(74) \left\{ \begin{aligned} s_{p_x} &= \sqrt{\frac{\Sigma v^2_{x_{p_j}}}{\text{no. pts. in tracking interval}}} & \quad u_{p_x} &= \frac{\Sigma v_{x_{p_{j+1}}} v_{x_{p_j}}}{\text{no. pts. in tracking interval}} \\ s_{p_y} &= \sqrt{\frac{\Sigma v^2_{y_{p_j}}}{\text{no. pts. in tracking interval}}} & \quad u_{p_y} &= \frac{\Sigma v_{y_{p_{j+1}}} v_{y_{p_j}}}{\text{no. pts. in tracking interval}} \\ r_{p_y} &= \sqrt{\frac{\Sigma v^2_{r_{p_j}}}{\text{no. pts. in tracking interval}}} & \quad u_{p_r} &= \frac{\Sigma v_{r_{p_{j+1}}} v_{r_{p_j}}}{\text{no. pts. in tracking interval}} \end{aligned} \right.$$

$$(75)\quad \begin{cases} \rho_{px} = u_{px}/s_{px}^2, \\ \rho_{py} = u_{py}/s_{py}^2 \\ \rho_{pr} = u_{pr}/s_{pr}^2 . \end{cases}$$

Repeat the general adjustment with s_{px}, s_{py}, s_{pr} replacing σ_{px}, σ_{py}, σ_{pr} and with the values of ρ_{px}, ρ_{py}, ρ_{pr} from (75) being used in place of values specified in input. Iterate to convergence. Recompute primary residuals, as in (70). In terms of primary residuals, compute secondary residuals as follows:

$$v'_{x_{p_j}} = v_{x_{p_j}} - \rho_{px} v_{x_{p_{j-1}}}$$

$$v'_{y_{p_j}} = v_{y_{p_j}} - \rho_{py} v_{y_{p_{j-1}}}$$

$$v'_{r_{p_j}} = v_{r_{p_j}} - \rho_{pr} v_{r_{p_{j-1}}} .$$

Compute rms of primary and secondary residuals for each tracking interval.

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PART 2. COMPUTER PROGRAM DEVELOPMENT

By

Jerry E. Trotter

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PART 2. COMPUTER PROGRAM DEVELOPMENT

1.0 INTRODUCTION*

The development of the Short Arc Geodetic Adjustment (SAGA) program has provided satellite geodesy with another very accurate reduction tool. This program, we believe, is one of the most advanced reduction programs yet developed for precise geodetic positioning by means of satellite observations. It can handle any combination of directional or ranging observations. Specifically, these are:

a. Optical (active and passive): PC-1000, MOTS, BAKER-NUNN, BC-4,

b. Electronic: Geodetic SECOR, GEOCEIVER, LASER.

Each observation parameter may be subject to systematic errors governed by an error model having either unknown coefficients or statistically constrained coefficients. In the optical case the angular elements of external orientation α, ω, κ may be subjected to slight adjustments that are consistent with their actual accuracies. The orientation accuracy is typically 0".5 of arc in α, ω and 2".0 in κ. The exercise of orientation error model constraints is particularly important in chopping shutter observations. A conventional reduction of a large quantity of chopping shutter observations per plate, with the existence of systematic error in orientation would lead to an unduly optimistic error propagation. In a chopping operation (as opposed to a flashing light), allowances should be made for uncertainties in inter-station timing synchronization. The inter-station timing bias simulates the case of non-synchronization between stations observing a common satellite pass. The electronic error model accomodates biases in zero set, timing, frequency (satellite oscillator), frequency (ground oscillator), frequency drift and refraction. SAGA will accept as many as 4 observation intervals from the various trackers. This allows observation drop out with a new zero setting of ranges and re-orientation of the optical tracker.

* The material presented in the first two sections of Part II was originally given in a paper at the American Geophysical Union (AGU) Meeting in Washington, D.C., in May 1969 as a joint effort by DBA and AFCRL (see Reference 6).

The error model adopted for ranging systems is of the form

$$\delta r = a_0 + a_1 \tau + a_2 \tau^2 + a_3 r + a_4 \dot{r} + a_5 \csc E$$

in which

$r, \dot{r}$ = range and range rate at time τ ($\tau = 0$ at epoch)

E = local elevation angle

$a_0, a_1, a_2, a_3, a_4, a_5$ = error coefficients accounting for systematic errors in zero set, frequency offset (between satellite and ground station oscillators), frequency drifts, frequency bias, timing bias and residual refraction.

The model was derived to apply specifically to Geoceiver observations but is applicable to ranging systems in general when appropriate constraints are placed on the error coefficients.

Systematic errors in optical observations are assumed to be governed by error models of the form:

$$\delta x = a_0 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 + a_5 f_5$$

$$\delta y = a_0 f_1' + a_2 f_2' + a_3 f_3' + a_4 f_4' + a_5 f_5'$$

where

$f_1 = -(c^2 + x^2)/c \qquad f_1' = xy/c$

$f_2 = xy/c \qquad f_2' = -(c^2 + y^2)/c$

$f_3 = y \qquad f_3' = -x$

$f_4 = x/c \qquad f_4' = y/c$

$f_5 = \dot{x} \qquad f_5' = \dot{y}$

in which

x, y = plate coordinates of satellite image

$\dot{x}, \dot{y}$ = rate of change of plate coordinates

c = nominal focal length of camera.

These coordinates are generally reconstructed from published angles (right ascension, declination) by a process of 'dummy camera projection', wherein the given angles are projected onto the plate of a fictitious camera aimed at the middle of the recorded arc. The optical error coefficients a_0 through a_8 account for the combined effects of biases in the angular elements of orientation of the camera, the elements of interior orientation (coordinates of principal point and focal length), and timing. They also account for any linear drift in the direction of the camera axis throughout the exposure.

To demonstrate the capabilities and the performance of SAGA, we reduced a network consisting of optical and ranging observations. These involved the PC-1000 (active and passive), MOTS, SECOR and the LASER. Twenty seven (27) orbits were chosen that provided a good geometrical relationship between observing stations and observed orbits for accurate station position determination (see Figure 1).

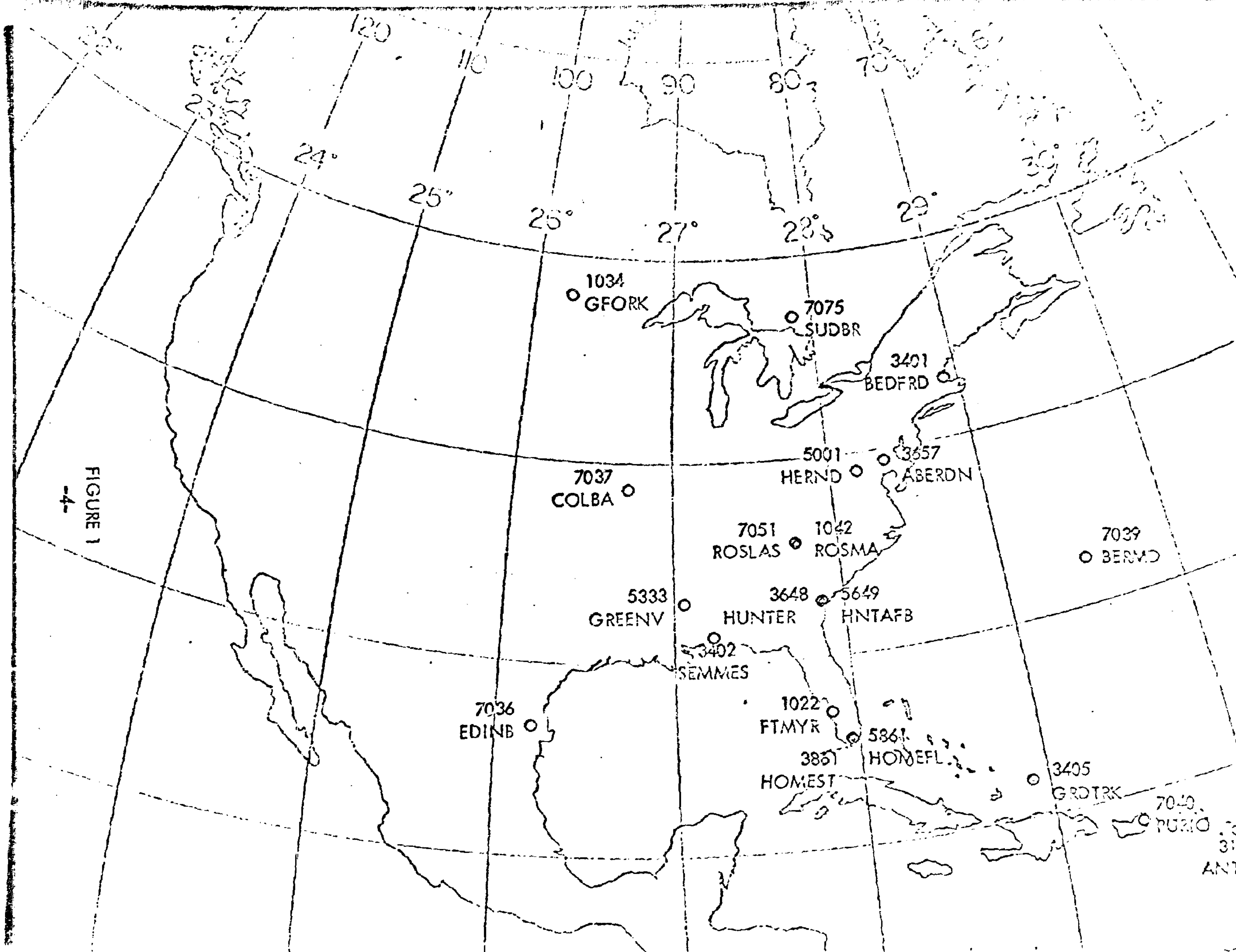

1034 GFORK
7075 SUDBR
3401 BEDFRD
5001 HERND
3657 ABERDN
7037 COLBA
7051 ROSLAS
1042 ROSMA
7039 BERMD
5333 GREENV
3648 HUNTER
5649 HNTAFB
3402 SEMMES
7036 EDINB
1022 FTMYR
5861 HOMEFL
3861 HOMEST
3405 GRDTRK
ANTI
120
110
100
90
80
70
24°
25°
26°
27°
28°
29°

FIGURE 1

2.0 RESULTS OF REDUCTIONS

Twenty stations participated in the complete network. The one sigma error of the network survey determination was typically 3 to 5 meters, depending upon the station geometry relative to the network and the amount of observations from the station. The error propagation is internal to the network with the origin at Hunter AFB.

The optical observations were assumed subject to orientation biases of 1".0 in α, ω and 2".0 in κ for all passes. The orbit parameters were completely relaxed to 1×10^4 meters in position, 5. meters/second in velocity. The range measurements were assumed to be subject to a systematic bias of 10 meters with a random noise of 1 meter. The random error of the optical observation was 5 microns in plate x and y coordinates. See Table I for all input errors.

The SECOR observations were made on orbits that had no optical observations. Four orbits were observed by four SECOR stations (5333, 5001, 5649, 5861) and one orbit was observed by three stations (5333, 5001, 5649). Matching of available SECOR observations with available optical measurements resulted in little strength from either. Therefore the five were selected that provided good geometry with the four SECOR stations. The SECOR stations were incorporated into the overall network by applying constraints between the SECOR and optical co-located stations (e.g. 3861, 5861, see Figure 2).

Laser range measurements from station 7051 were obtained on four passes in conjunction with optical observations. Optical station 1042 and station 7051 are again treated as co-located stations by constraining the direction and distance between them.

Table II reflects the strength of each pass into the overall solution. For instance, orbits twenty one and twenty two are very weak in orbit determination since only two stations observed. The accuracy of the orbit determination reflects the degree of contribution of the pass to overall surveys. The accuracy of the orbit position (X, Y, Z) is largely dependent on the intersection angle of station observations and the accuracy of the measurements. The determination of velocity components improves significantly as the length of the data span observed increases.

TABLE 1

Station Participation

Orbit No.	Participating Stations	Intervals	# Observed Sets*
9	1034, 7039, 7037, 7075	4	9
2	1042, 7037, 7040, 7039, 1022, 7075	3	12
35	7039, 7040, 1042, 7051	2	6
5	7040, 7036, 7039, 1022, 7051	2	6
36	7036, 1022, 7075, 7051	2	4
37	7039, 7037, 7075, 7040, 7051	4	9
1	7040, 7037, 7039	2	4
3	7040, 7037, 1034, 7036, 7039, 7075	3	11
8	7036, 7039, 7037, 1022, 1042, 7075	3	9
10	7040, 1042, 7039	3	6
11	7039, 3405, 3402, 3861	2	5
12	7039, 1022, 7040, 1042, 3861	3	8
13	7036, 7037, 7039	2	4
4	7037, 7036, 7039, 7075	4	10
20	3657, 3861, 3401	1	3
21	3401, 3405	2	2
22	3401, 3106	2	2
23	3657, 3405, 3648	2	5
24	3657, 3106, 3648	2	4
25	3657, 3405, 3402	1	3
26	3401, 3402, 3648, 3106	2	5
27	3402, 3401, 3657, 3106	2	4
40	5001, 5649, 5333, 5861	1	4
39	5001, 5649, 5333, 5861	1	4
42	5001, 5649, 5333, 5861	1	4
41	5001, 5649, 5333	1	3
43	5001, 5649, 5333, 5861	1	4
	Total number of observation sets		131

*The number of observed sets represent the total number of camera photos and intervals of range observation for each pass. Each flash sequence represents one interval.

TABLE 2

A Priori Constraints on Error Parameters

Station No.	Input Sigmas Latitude (sec.)	Longitude (sec.)	Height (meters)
5001	0.200	0.200	5.00
5333	0.200	0.200	5.00
5649	0.001	0.001	0.10
5861	0.200	0.200	5.00
7051	0.200	0.200	5.00
3405	0.300	0.300	8.00
3402	0.200	0.200	5.00
3357	0.200	0.200	5.00
2106	0.350	0.350	8.00
3861	0.200	0.200	5.00
3401	0.200	0.200	5.00
7040	0.350	0.350	8.00
1022	0.200	0.200	5.00
1034	0.200	0.200	5.00
1042	0.200	0.200	5.00
7037	0.200	0.200	5.00
7036	0.200	0.200	5.00
7039	0.800	0.800	25.00
7075	0.200	0.200	5.00
3648	0.001	0.001	0.10

Observations

Optical:

Orientation:

σ_α = 1. Arc seconds

σ_ω = 1.

σ_κ = 2.

Timing (Inter-station):

σ_t = 1×10^{-4} (active)

σ_t = 3 milliseconds (passive)

Measurement:

σ_x = 5 microns } Plate coordinates
σ_y = 5 microns }

σ_c = 10 microns Focal length

σ_t = 1×10^{-4} Random timing

Electronic:

Zero set, σ_{s_o} = 10 meters
Random Range, σ_r = 1 meter
Timing, σ_t = 1×10^{-4}

Input Error of Initial Conditions:

Position, Velocity

$\sigma_x = \sigma_y = \sigma_z$ 1×10^{4} meter
$\sigma_{\dot{x}} = \sigma_{\dot{y}} = \sigma_{\dot{z}}$ meters/seconds

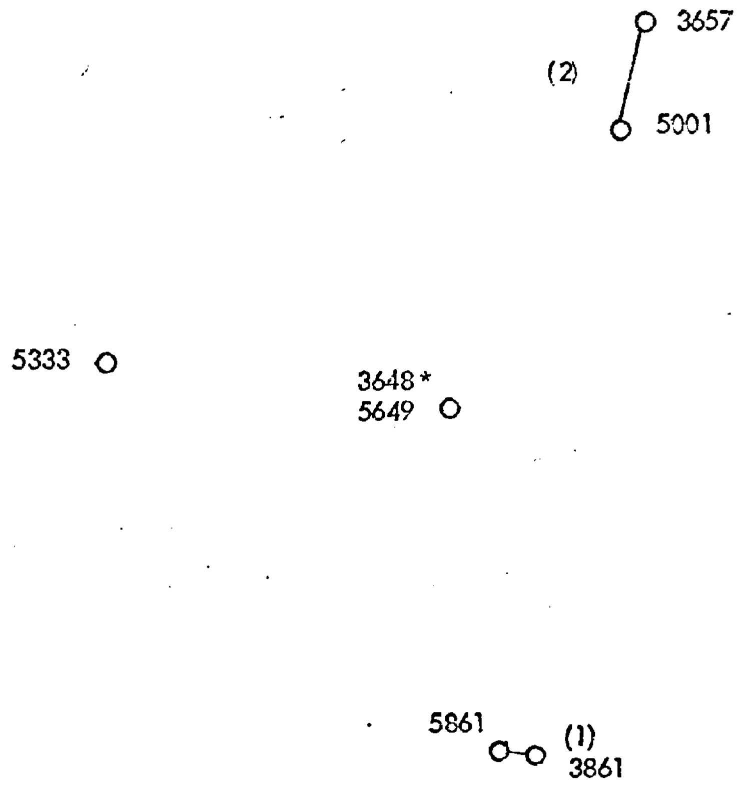
TABLE 3
Standard Deviation of Recovered Orbital Elements
(Meters and Meters/Second)

Pass No.	σ_x	σ_y	σ_z	$\sigma_{\dot{x}}$	$\sigma_{\dot{y}}$	$\sigma_{\dot{z}}$
9	14	14	37	.027	.031	.054
2	9	12	7	.029	.033	.026
35	8	9	8	.021	.024	.021
5	34	28	34	.032	.026	.040
36	6	8	7	.031	.022	.020
37	7	8	7	.024	.023	.027
1	69	272	279	.158	.411	.468
3	11	14	8	.022	.028	.021
8	12	19	14	.035	.054	.040
10	15	17	19	.035	.038	.064
11	27	30	12	.067	.123	.132
12	14	7	8	.037	.036	.044
13	29	24	44	.088	.103	.101
4	10	17	10	.019	.031	.021
20	7	13	13	.518	.887	.944
21	29	92	184	.397	.394	1.097
22	128	140	223	.786	.936	.906
23	20	9	19	.034	.038	.043
24	78	23	20	.347	.044	.140
25	15	18	23	.838	1.126	1.634
26	11	15	13	.050	.083	.115
27	5	11	8	.196	.321	.432
40	29	31	12	.059	.094	.052
39	30	19	18	.111	.078	.062
42	46	10	22	.048	.036	.029
41	25	54	10	.061	.105	.070
43	17	36	26	.037	.035	.054

TABLE 4

Station Coordinate Adjustments (Meters)

Station	Corrections			Standard Error		
	ΔX	ΔY	ΔZ	σ_X	σ_Y	σ_Z
5001	2.59	-3.18	-5.03	3.4	4.5	3.2
5333	5.77	-0.40	-12.16	3.9	4.8	4.7
5649	0.0	0.0	0.0	0.03	0.09	0.06
5861	8.51	-8.50	-5.02	3.3	3.3	3.2
7051	-7.99	-9.39	-8.11	4.1	3.9	3.5
3405	-2.0	-9.66	-15.32	7.2	7.4	7.9
3402	10.41	-0.42	-2.77	4.9	4.9	5.4
3657	-3.02	6.29	-4.93	3.1	4.3	3.2
3106	15.57	9.76	5.00	9.2	8.0	9.7
3361	8.48	-8.56	-5.00	3.3	3.3	3.2
3401	-7.34	-3.98	4.37	4.0	5.0	4.8
7040	-2.42	23.81	-13.91	6.2	6.5	6.9
1022	-1.16	2.03	-2.72	4.4	4.3	4.8
1034	-1.81	-6.31	0.69	4.0	5.0	5.2
1042	2.94	2.34	2.74	4.0	4.0	3.5
7037	-2.28	-0.68	-10.06	3.7	4.1	4.1
7036	13.77	-13.58	3.58	4.7	4.6	4.7
7039	18.71	5.74	6.81	5.5	6.0	5.5
7075	-0.60	5.53	2.07	3.5	4.4	4.3
3648	0.0	0.0	0.0	0.03	0.09	0.06



* origin station

Constraints

	Distance (meters)	Direction (arc sec)
(1)	.2	1.0
(2)	1.0	.5

Figure 2

Constrained Relationships between SECOR and Optical Stations

The total corrections (Table IV) to survey and the propagated errors are consistent with the observation network with the origin at Hunter AFB. The propagated errors reflect the internal network accuracies and illustrate the strength of this particular solution. The recovery of some stations was weaker than others due to the limited observations from the station. Bermuda was a priority factor in selecting orbits, therefore the network provided strong geometry for the determination of its coordinates. Bermuda (7039) participated in thirteen (13) passes which involved twenty four (24) plates (or 24 different flash sequences). Antigua participated in only four different passes with a total of 5 plates. The position recovery for Antigua was nearly twice that for Bermuda. On the other hand the input error for Antigua was much smaller than that for Bermuda (meaning the a priori information was better). The improvement of the recovered coordinates over the a priori information was not as significant for Antigua. Table III lists the observing station for each pass.

A typical recovered bias in optical orientation α, ω, and $\times$ was .2 arc seconds. The largest recovered bias was .8 arc seconds. Recovered biases in laser observations ranged from 4 to 10 meters and SECOR biases ranged from 6 to 27 meters. The random noise was approximately 1 to 2 meters in both LASER and SECOR measurements.

This program accommodates adjustments to the center of mass. The results presented were obtained with the center of mass held fixed. As a preliminary experiment an adjustment was made with the center of mass relaxed to 50 meters in X, Y, Z. The corrections to the coordinates of the center of mass were 10, 35 and -27 in X, Y and Z respectively with standard deviations of 39 meters. The change in station surveys was negligible. This experiment was made to demonstrate the programs capabilities in recovering the center of mass. The network is concentrated on a small section of the earths surface and the geometry with respect to the center of mass was very weak. Further studies will be undertaken with emphasis placed on orbital geometry and observing station locations relative to the center of mass.

Conclusions

In view of the various instrumentations and observing modes used in this reduction, it is apparent that this program could be used with any type optical or ranging system presently employed in global tracking. The Geoceiver was given some special consideration in the error model of range measuring type systems. Terms for frequency biases and frequency drift were designed to accommodate errors introduced by the satellite and ground oscillators.

This presentation is intended to primarily demonstrate the capabilities of the SAGA reduction program. It is not our intention to imply that the results should lead to a change in position of Bermuda or any other station in the network. But the corrections and accuracies in the subject solution are meaningful with respect to this network. The accuracy of 5 to 6 meters recovery of the Bermuda coordinates clearly illustrates the potential of recovering the surveys of other stations that have large survey errors. The reduction demonstrates the impressive potential of SAGA as a tool for establishing continental and inter-continental surveys to a high degree of accuracy.

3.0 PROGRAM DESCRIPTION

The SAGA program is a multi-orbital reduction program designed to recovery station geodetic coordinates and bias parameters associated with the measuring system being employed. As many as four zero settings of the defined orbit may be used. This provides the capability of observing as many as four different segment combinations of the orbit by the various trackers. Optical and ranging observations are accepted and these are specifically PC-1000, BC-4, MOTS, BAKER-NUNNS, SECOR, LASER, and GEOCEIVER.

Input formats are primarily the same as the GEOS format except in special cases in which the data has been processed through a data prep program (see Appendix D) for special corrections. This is a program option which was included to provide program simplicity and more importantly to provide SAGA with consistent and pre-edited data.

SAGA consists of six major programs which are directed by a control program. The first two (MASTER and PREP) read all input data and store them into disk files or tape for use in the iteration cycle. They also compute the station covariance matrix and baseline constraints. The orbit integrator, which is part of the iteration cycle, performs orbit integration from the time of the initial conditions to the desired epoch. At the time of epoch it updates the position, velocity and time on the first iteration. Subsequent iterations need only to be expanded on the corrected orbit and no further integration performed. The user should provide initial conditions as near the desired epoch as possible to prevent error build up which leads to additional iterations to acquire convergence. NORMAL forms the normal equations. SOLVE solves for survey correction. OUTER solves for orbit and observation biases and updates the master file.

Five iterations are the maximum number of iterations before cutoff. If convergence is not obtained before the cutoff point the data is probably unstable and may be diverging. At this time one should analyze the data characteristics.

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4.0 SUBROUTINE DESCRIPTION

This section gives a brief description of subroutines and their primary mathematical operation. The calling sequence of each is defined as follow

CLEAR (X, N)

This subroutine clears an array X of size N.

X Array to be cleared

N Number of elements in X

AUGVRT (A, NR, NC, ND)

AUGVRT replaces the matrix A by A inverse augmented by the solution matrix.

A The matrix to be inverted

NR Row dimension of A to be inverted

NC Column dimension of A to be inverted

ND Actual dimension of A

MATMPY (A, NRA, NCA, B, NRB, NCB, C, M1, M2)

MATMPY will multiply A and B and store into C. The options of M1 provide any allowable transpose combination and M1 allows such storage combinations as the simple product, negative product, sum into previous C (positive or negative).

A First array to be multiplied

NRA Rows of A

RCA Columns of A

B Second array

NRB Rows of B

NCB Columns of B

C Storage array of product AB

M1 Transpose option

M2 Action of product R to storage array C

UPDATE (ICN, KTR, EPS, DEL, XPO, YPO, ZPO, XOT, YOT, ZOT)

This program updates the matrizant for a new step in time.

ICN	Makes decision on to integrate position and velocity
KTR	Number of terms to use in series
EPS	Truncation error limit
DEL	Valid step size
XPO, YPO, ZPO	Input position
XOT, YOT, ZOT	Output position

ZER (A, N, M)

ZER zeros the array A for N times M values starting at location A(1).

A	Array to be set to zero
M	Rows of A
N	Columns of A

ATRACT (L, K, B, S, IOT, NS, NROW)

This routine will disect array B in 3x3 matrices C (3x3) and write a file on tape or disk identified by station ID from array L.

L	Array containing identification of disected 3x3 arrays
K	Number of 3x3 matrices along diagonal of B
B	Array containing K squared 3x3 matrices
S	Array containing K 3x1 matrices
IOT	Unit number of tape or disk to be written on
NS	Dummy integer
NROW	Rank of desired partitioned size array

DUMMY (CAST, OR, A, E, TP, IC, IS, PH, CH, HE, IDNT, P, Q)

This program simulates a dummy camera projection of right ascension and declinations to plate coordinates.

GAST	Greenwich sidereal time
OR	Output orientation matrix
A	Array of plate X coordinates
E	Array of plate Y coordinates
TP	Time of i th X, Y coordinate
IC	Number of coordinates read
IS	Station identification number
PH	Array of station latitudes
CH	Array of station longitude
HE	Array of station heights
IDNT	Array of station identification numbers
P	Latitude of observing station
Q	Longitude of observing station

COVAR (SLA, SLAM, SH, R, A, ECC, SIGMA)

This routine computes the covariance matrix of a station in geocentric coordinates given geodetic error in latitude, longitude and height.

SLA	Latitude of station
SLAM	Longitude of station
SH	Height of station
R	Output covariant matrix (3,3)
A	Semi-major axis of earth
ECC	Eccentricity of earth
SIGMA	Input errors in latitude, longitude and height

EXTRAC (L, K, B, CC, NS)

This subroutine extracts 3x3 matrices from array B and writes them on unit 3.

L	Array containing position of 3x3 matrices in B that are going on unit 3
K	Number of 3x3 matrices along the diagonal of B

B	Array being disected
CC	State vector to be disected into 3x1 matrices
NS	Size of submatrix.

DRIVER (NN, IU, TM, GH, DT, XOT, VEM)

This program evaluates the coefficients for position, velocity and matrizant at time TM.

NN	If NN = 1 read tape unit IU for coefficients and orbit parameters at epoch, otherwise expand only about epoch
IU	Tape unit to be read
TM	Time of observation
GH	Corrected output time from epoch
DT	Increment in time that coefficients are good
XOT	Output position and velocity
VEM	Output matrizant

MATRUP (KTR, DEL, UVM, UVO)

Subroutine to update matrizant with respect to time.

KTR	Number of terms in series
DEL	Increment in time from epoch
UVM	Coefficient matrix
UVO	Updated matrizant

CRV (AX, ECC, VC, IDNT, EPS, ID, IPR)

CRV updates station geocentric coordinates from corrections in previous iteration.

AX	Semi-major axis of earth
ECC	Eccentricity of earth
VC	Array containing station X, Y, Z
IDNT	Array of identification numbers

EPS	Array of station corrections
ID	ID of station match
IPR	Flag set if station has already been outputed

LLH (AX, ESQR, P, Q, R, OP, OL, OH)

Conv geocentric Cartesian coordinates to geodetic coordinates.

AX	Semi-major axis of earth
ESQR	Eccentricity squared of earth
P	Geocentric Z
Q	Geocentric X
R	Geocentric Y
OP	Output latitude
OL	Output longitude
OH	Output height

ATANN (X, Y)

This is a function that computes the angle between the vectors X and Y.

X	First vector
Y	Second vector

DEG (AN, I, J, S)

DEG converts an angle AN to degrees, minutes and seconds.

AN	The angle in radians
I	Output degrees
J	Output minutes
S	Output seconds

GEOC (XL, YL, HT, A, ES, V)

GEOC converts geodetic position (Φ, λ, h) to geocentric coordinates (X, Y, Z).

XL	Latitude (radians)
YL	Longitude (radians)
HT	Height (meters)
A	Semi-major axis of earth (meters)
ES	Eccentricity of earth squared
V	Returned array containing X, Y, Z

ERCOE (EPP, NID, K, L, J)

ERCOE outputs observation bias corrections of electronic error model terms.

EPP	Array containing bias corrections
NID	Array containing station identifications
K	Defines observation interval
L	Station identification
J	Index defining station in pass

ERMOD (EPP, NID, K, L, J)

ERMOD outputs observation bias corrections of optical error model terms.

EPP	Array containing bias corrections
NID	Array containing station identifications
K	Defines observation interval
L	Station identification
J	Index defining station in pass

EXPAND (XPO, YPO, ZPO, CNM, SNM, LCT, ICT, UMT, VMT, CTB, CTT, ERD, XMU, ALF, OMG, ECC, NTE, KTR, KDR, NHT, CDC, CTW, KEY, DMT, KRG)

XPO, YPO, ZPO	Power series for position (X, Y, Z)
CNM, SNM	Potential coefficient
LCT, ICT	Control tables
UMT	Expansion series
VMT	Expansion series
CTB, CTT	Coefficient tables
ERD	Radius of earth
XMU	Gravity constant
ALF	Greenwich hour angle
OMG	Rotation rate of earth
ECC	Eccentricity squared
NTE	Tables control parameter
KTR	Number of terms in series
KDR	Number of terms in drag
NHT	Zero
CDC	Ballistic coefficient
CTW	Drag time
KEY	Integration control constant
DMT	Dummy array
KRG	Defines the number of terms in series.

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5.0 INPUT/OUTPUT INSTRUCTIONS

A. Input

This section will describe the program input parameters and illustrate the output formats. Input will be defined as specification cards and a complete orbit set. A run consisting of more than one orbit is made by simply stating the number of orbits on the first specification card. A complete set of N orbits is illustrated in Figure 1.

Baseline Constraints

Card No.	Name	Field	Columns	Units	Description
1	IP	I4	1- 4		Station ID (1st station)
	IQ	I4	5- 8		Station ID (2nd station)
	IS	I2	9-10		$\neq 0$, read linear constraints
	AL	F9.0	11-19	Degrees	A priori azimuth
	EL	F9.0	20-28	Degrees	A priori elevation
	RA	F9.0	29-36	Meters	A priori range
	S(1)	F7.0	38-45	Arc sec	Azimuth sigma
	S(2)	F7.0	46-53	Arc sec	Elevation sigma
	S(3)	F7.0	54-61	Meters	Range sigma
	U1	F12.0	62-73		Linear error if $IS \neq 0$
	SU	F7.0	74-80		Origin error
2*	XD	F12.0	25-36	Degrees	Station latitude
	YD	F12.0	37-48	Degrees	Station longitude
	HT	F12.0	49-60	Meters	Station height
3*	AX	F12.0	1-12	Meters	Earth axis
	ECC	F12.0	13-24		Eccentricity squared
	XP	F12.0	25-36	Degrees	Station latitude
	YP	F12.0	37-48	Degrees	Station longitude
	HP	F12.0	49-60	Meters	Station height

* Station position for the two baseline stations involved.

Card No.	Name	Field	Columns	Units	Description
If IS $\neq 0$					
4	COE(1)	F11.0	1-11		Linear coefficients between station pair
	:	:	:		
	COE(7)	F11.0	67-77		

Repeat the above cards for each baseline desired, then follow the last baseline set with a blank card.

5	BLANK				

Specification Cards

Card No.	Name	Field	Columns	Units	Description
1	NPASS	F13.0	1-13		Number of passes
	ECC	F13.0	14-26		Eccentricity squared
	AX	F13.0	27-39	Meters	Semi-major axis
	GR	F13.0	40-52	Meters^3/Sec^2	Gravitation constant
	ROT	F13.0	53-65	Radians/Sec	Rotation rate of earth
2A	IDNT	I7	1- 7		Station ID
	G1	F7.0	8-14	Degrees	Latitude
	G2	F7.0	15-21	Minutes	
	G3	F7.0	22-28	Seconds	
	G4	F7.0	29-35	Degrees	Longitude (West)
	G5	F7.0	36-42	Minutes	
	G6	F7.0	43-49	Seconds	
	G7	F7.0	50-56	Meters	Height (above spheroid)
	CM(1)	F7.0	57-63	Seconds	Sigma latitude
	CM(2)	F7.0	64-71	Seconds	Sigma longitude
	CM(3)	F7.0	72-78	Meters	Sigma height

Card No.	Name	Field	Columns	Units	Description
2B	2B is the same as 2A for the next station. These are station cards. Follow the last station card with a blank.				
2C	Blank				
2D	σ_X	F13.0	1-13	Meters	Sigma of center of mass
	σ_Y	F13.0	14-26	Meters	
	σ_Z	F13.0	27-39	Meters	
3A	ST1(1)	F11.0	1-11	Meters	Input sigma of plate x
	2		12-22	Unitless	Correlation coefficient
	3		23-33	Meters	Sigma of plate y
	4		34-44	Unitless	Correlation coefficient
	5		45-55	Seconds	Sigma of time
	6		56-66	Meters	Sigma of range
	7		67-77	Unitless	Correlation coefficient
3B & 3C	3B and 3C are the second and third alternate schedules of error inputs for plate or pass data. 3A is the standard schedule of error input. Cards 3 and 4 are for optical observations.				
4A	ST21(1)	F13.0	1-13	Radians	Orientation sigma α, ω, κ
	2		14-26	Radians	
	3		27-39	Radians	
	4		40-52	Meters	Focal length sigma
	5		53-65	Seconds	Interstation timing sigma
4B & 4C	4B and 4C are the alternate schedules of sigmas for station measurements.				
5A	ST31(1)	F13.0	1-13	Meters	Error coefficients of electronic error model
	2		14-26	Meters	
	3		27-39	Meters	
	4		40-52	Radians	
	5		53-65	Seconds	
	6		66-78	Radians	

Card No.	Name	Field	Columns	Units	Description
5B & 5C	5B and 5C are alternate schedules for range observations.				
6A	ST41(1)	F13.0	1-13	Meters	Sigmas of orbital initial conditions (position)
	2		14-26	Meters	
	3		27-39	Meters	
	4		40-52	Meters/Sec	Sigmas of orbital initial conditions (velocity)
	5		53-65	Meters/Sec	
	6		66-78	Meters/Sec	
6B & 6C	6B and 6C are alternate schedules of orbit sigmas.				

Orbit Input

Each orbit will be set up as shown below.

1	S1	7.0	Hours	Time of initial conditions
	S2	7.0	Minutes	
	S3	12.0	Seconds	
	C1	7.0	Hours	Time of desired epoch
	C2	7.0	Minutes	
	C3	12.0	Seconds	
	P1	7.0	Hours	Hour angle of Greenwich for zero hour of day of initial conditions
	P2	7.0	Minutes	
	P3	12.0	Seconds	

Card No.	Name	Field	Columns	Units	Description
2	IIN(1)	F13.0	1-13	Meters	Initial position
	2		14-26		
	3		27-39		
	4		40-52	Meters/Sec	Initial velocity
	5		53-65		
	6		66-78		
3	IDPAS	I5	1- 5		Pass identification
	IORB	I5	6-10		Selects orbit sigma schedule
	NSTA	I5	11-15		Number of stations in pass
	NINT	I5	16-20		Number of intervals observed
4*	FLEA	F12.0	1-12	Meters	Focal length if optical
	ITYPE	I8	13-20		- is electronic, + is optical
	IS	I8	21-28		Selects station sigma schedule
	IB	I8	29-36		Selects optical plate schedule
	NP	I8	37-44		Number of intervals observed
	NID	I8	45-52		Identification of interval observed

* Each station observing will have a card defined by 4. These cards define all data information for the particular station.

Card No.	Name	Field	Columns	Units	Description
5	INT	I4	1- 4		Number of total intervals observed
	HR	F10.4	5-14	Hours	Start time of observations
	MIN	F10.4	15-24	Minutes	
	SEC	F10.4	25-34	Seconds	
	XV	F10.4	35-44	Hours	Stop time of observations
	XN	F10.4	45-54	Minutes	
	XM	F10.4	55-64	Seconds	

The following cards are observation cards and will follow the same order as the station data definition cards 4. Let 6 represent an optical observation set and 7 electronic. A blank card follows each set of observations.

Card No.	Name	Field	Columns	Units	Description
6A	GEOS formatted data in right ascension and declination				
7A	GEOS formatted data in range				
6B	IID	I7	1- 7		Station identification number
	T(I)	F13.0	8-20	Seconds	Time of observation
	X(I)	F13.0	24-28	Meters	Plate X coordinate
	Y(I)	F13.0	29-46	Seconds	Plate Y coordinate
7B	IID	I7	1- 7		Station identification number
	T(I)	F17.0	8-24	Seconds	Time of observation
	RANGE	F17.0	25-41	Meters	Range measurement

B. Output

Program PREP outputs the primary input control parameters which define the error parameters and survey. This provides the user with later references to the network characteristics, participating stations and apriori errors of the particular computer run. Other output is adjusted parameters, residuals, standard deviations of recovered parameters, correction to all adjustable parameters and total corrections to survey parameters.

For each orbit in the solution the following output is written.

1) Residuals for each observation for each station.
2) Sigma (standard deviation) of recovered timing and orientation biases for optical measurements.

3) Sigma of recovered timing biases and range error model coefficients for range measurements.

4) Sigma of recovered orbit position and velocity.

5) Corrections to survey.

6) Sigma of survey corrections.

7) Corrections to position and velocity coordinates.

8) Corrections to measurement parameters.

9) Total corrections to survey.

10) Final survey in latitude, longitude, and height.

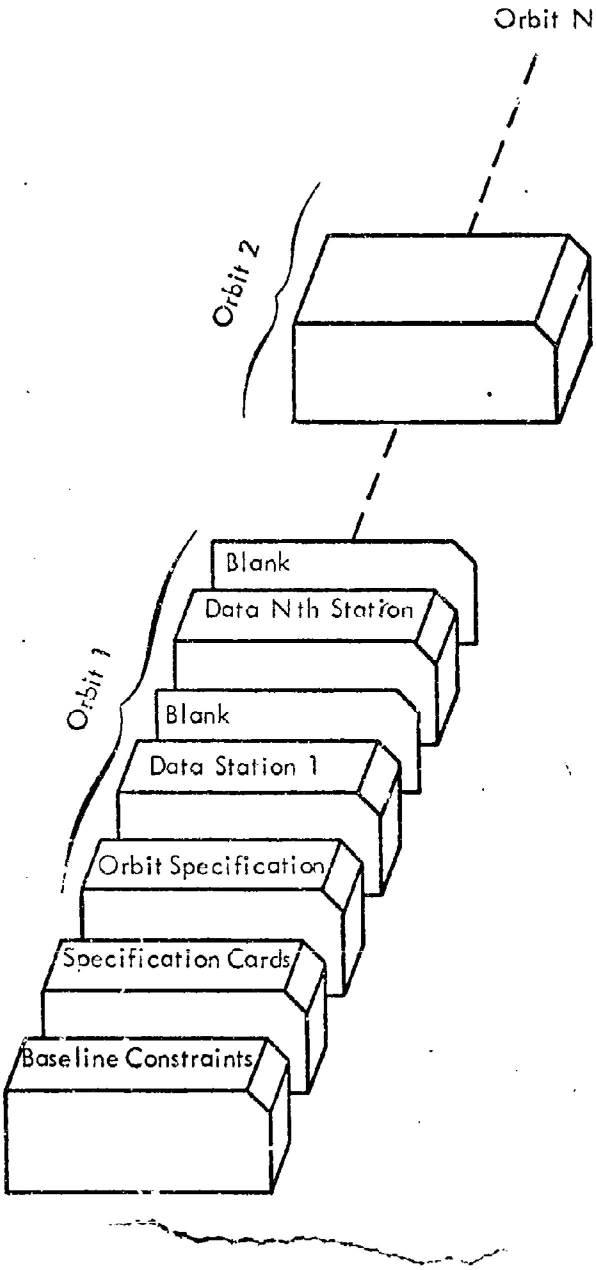

FIGURE 1. DATA SET

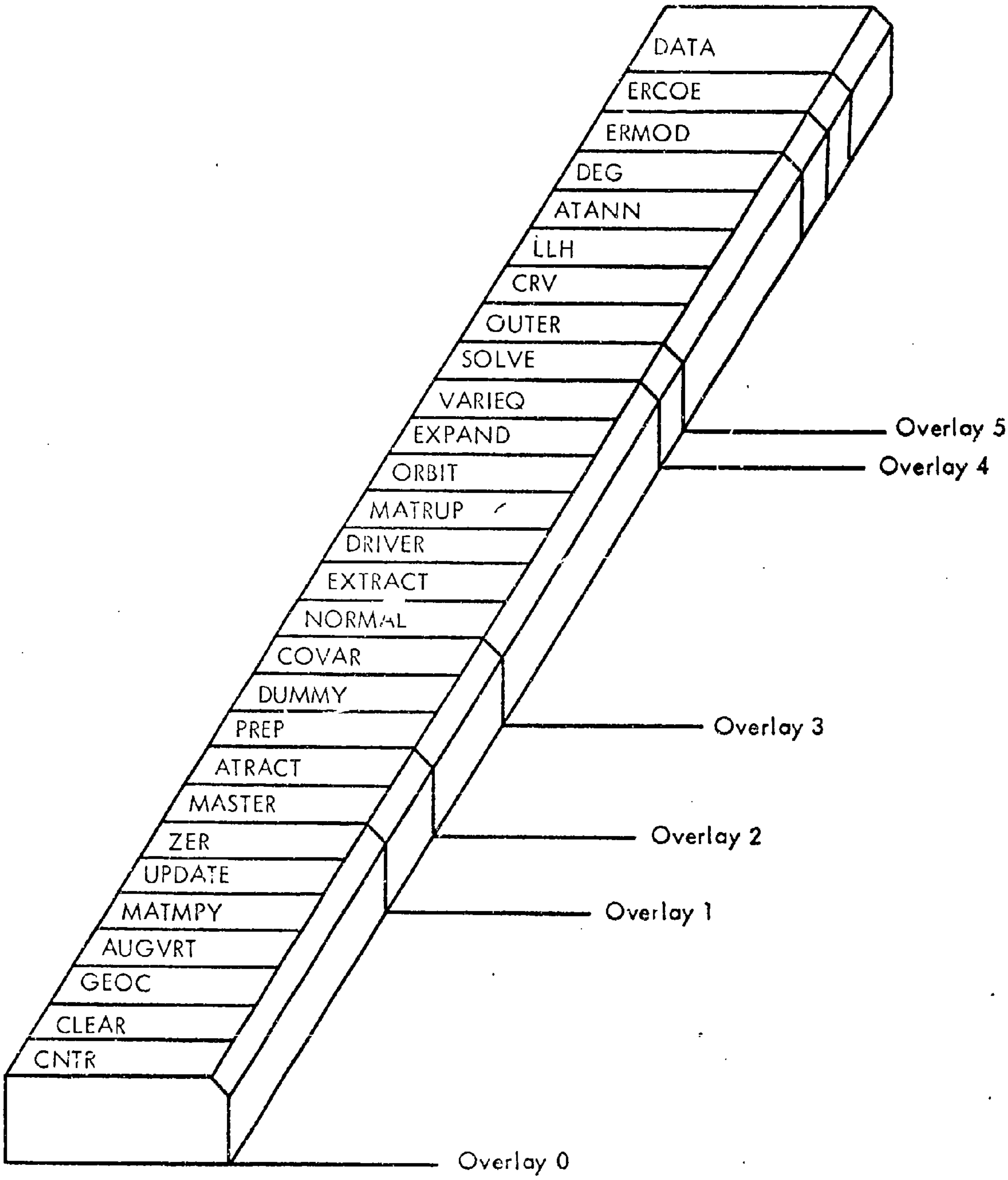

FIGURE 2. DECK SETUP

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6.0 GLOSSARY OF TERMS

A. Program Constants

Program Name	Quantity	Description
RAD	.01745329252	Radian to degrees conversion
GR	Input(optional)	Gravitation constant
ROT	Input(optional)	Earth rotation rate (meters^3/second^2)
AXIS	Input(optional)	Earth semi-major axis
ECC	Input(optional)	Earth eccentricity squared
CNM, SNM	Smithsonian standard earth model for 1966 through M, N = 4, 4.	

B. Symbol Correlation

Program Name	Math Symbol	Description
ROT	ψ	Earth rotation rate
GR	μ_e	Earth gravitation constant
STD(1)	σ_x	Standard error in plate x
STD(2)	ρ_x	Correlation coefficient in plate x
STD(3)	σ_y	Standard error in plate y
STD(4)	ρ_y	Correlation coefficient in plate y
STD(5)	σ_τ	Standard error in timing
STD(6)	σ_r	Standard error in range
STD(7)	ρ_τ	Correlation coefficient of range
STER(1)	$\sigma\alpha$	Error in orientation
STER(2)	$\sigma\omega$	Error in orientation
STER(3)	$\sigma\varkappa$	Error in orientation
STER(4)	σ_c	Error in focal length
STER(5)	σ_t	Interstation timing bias
STEL(1)	σ_{a_0}	Zero set error constraint
STEL(2)	σ_{a_1}	Interstation timing bias error constraint
STEL(3)	σ_{a_2}	Frequency bias constraint (satellite oscillator)
STEL(4)	σ_{a_3}	Frequency bias constraint (ground oscillator)
STEL(5)	σ_{a_4}	Frequency drift constraint
STEL(6)	σ_{a_5}	Residual refraction error constraint

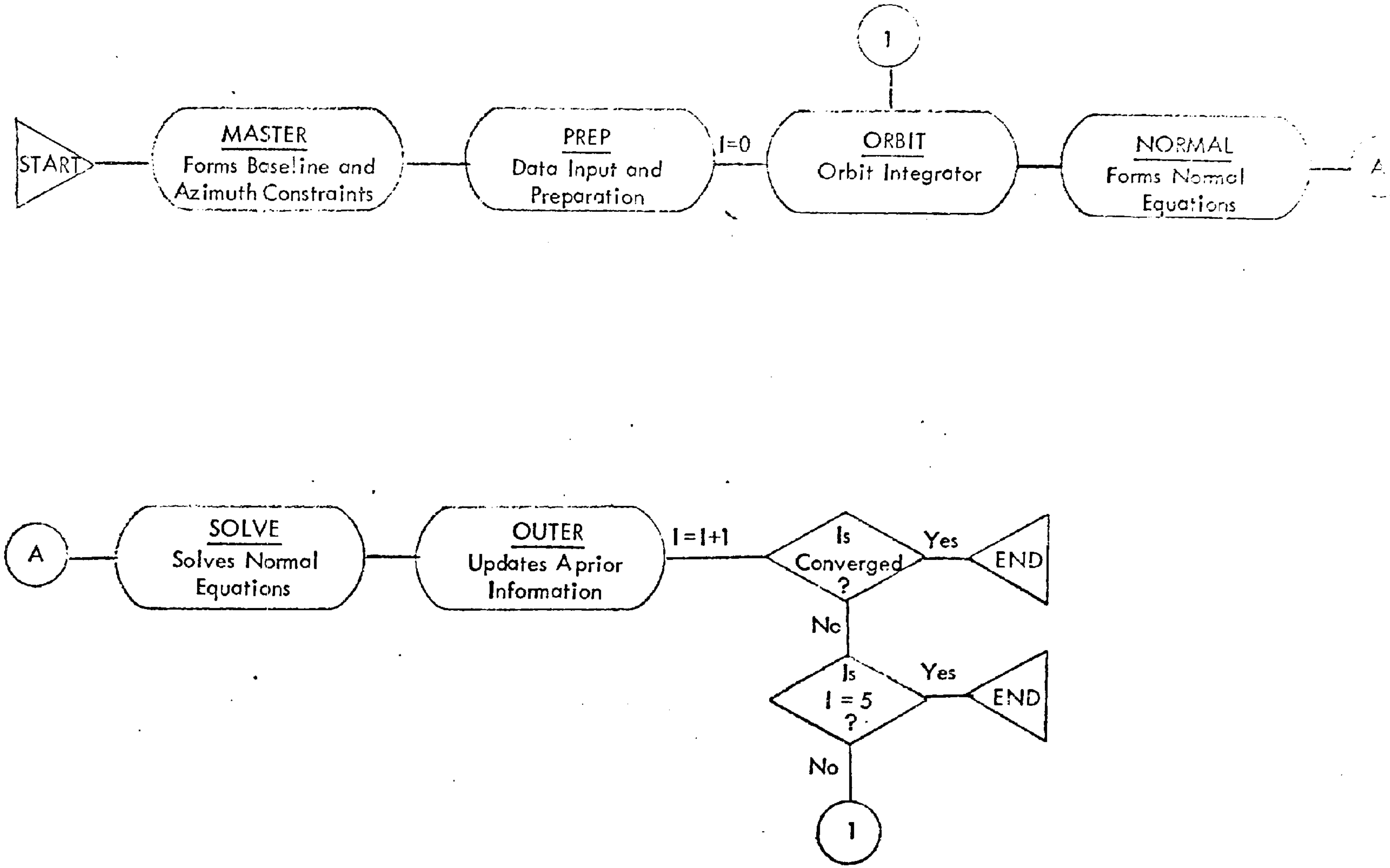
Program Name	Math Symbol	Description
ORB(1)	σ_x	Standard error of position and velocity
ORB(2)	σ_y	
ORB(3)	σ_z	
ORB(4)	$\sigma_{\dot{x}}$	
ORB(5)	$\sigma_{\dot{y}}$	
ORB(6)	$\sigma_{\dot{z}}$	
R1	R_j	Rotation matrix (inertial to earth fixed)
XE	X_j	Orbit position for the j th observation (earth fixed)
YE	Y_j	
ZE	Z_j	
X	X_i^c	Earth fixed coordinates of the i th station
Y	Y_i^c	
Z	Z_i^c	
XP	x_j^{oo}	Computed plate coordinates
YP	y_j^{oo}	
XJ	x_j^o	Measured plate coordinates
YJ	y_j^o	
TJ	t_j	Time of j th measurement
RJ	r_j^{oo}	Computed range
XJ	r_j^o	Measured range
DR	Δr_j	Refraction correction to range
WDD	$\ddot{W}_{ik}$	A priori weight matrix

Program Name	Math Symbol	Description
S	$\hat{\delta}$	Station correction vector
EPO	$\dot{\delta}_k$	Orbit correction vector
EPP	$\ddot{\delta}_{tk}$	Correction vector to error parameters
NSTA	m_k	Number of stations in k th pass
NPASS	K	Number of passes in network
IC	J	Number of observations
XDI	$\dot{X}$	Earth fixed velocity
YDI	$\dot{Y}$	
ZDI	$\dot{Z}$	
VK	Ω	Coefficients of matrizant polynomials

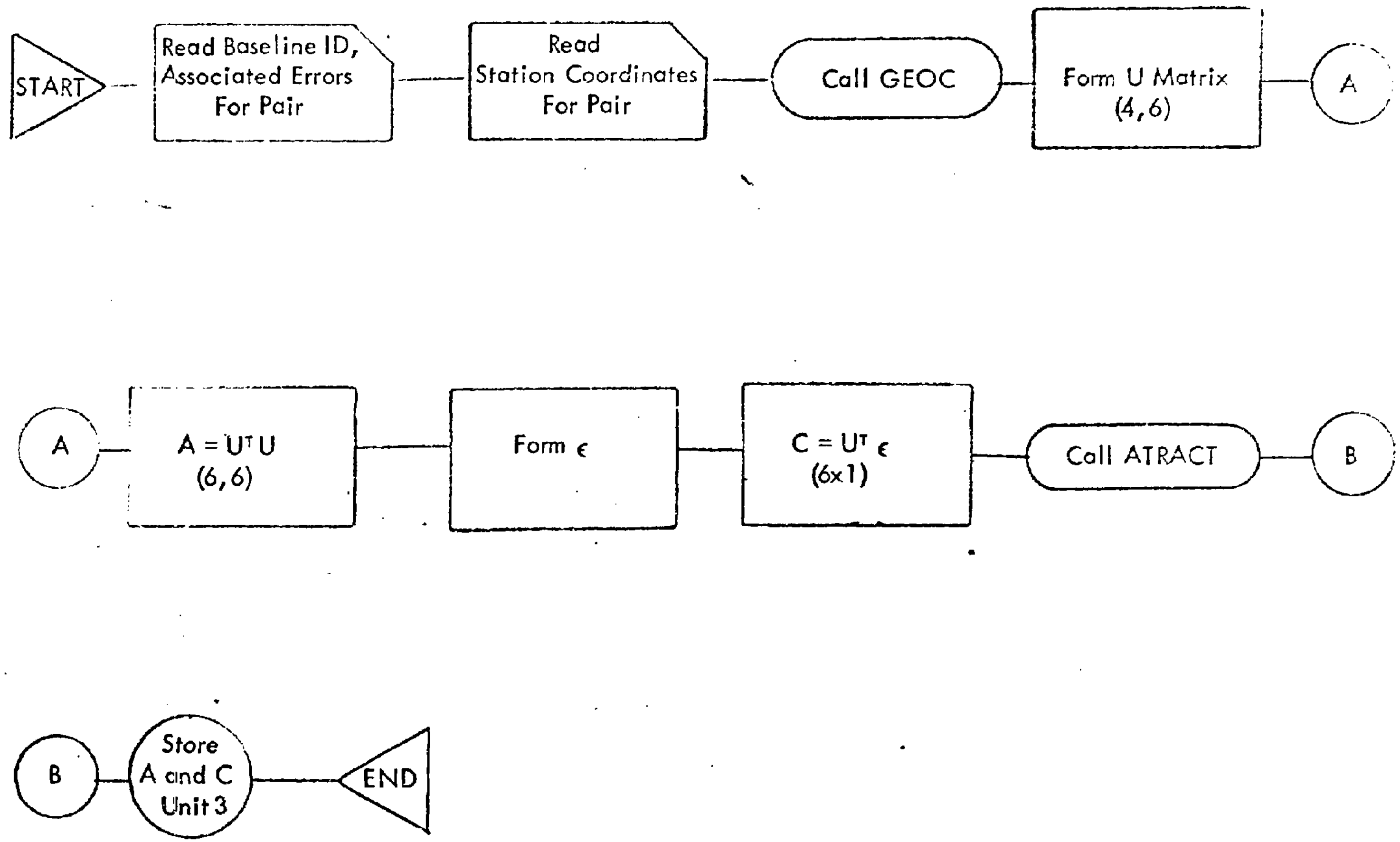
page intentionally blank.

7.0 FLOW CHARTS

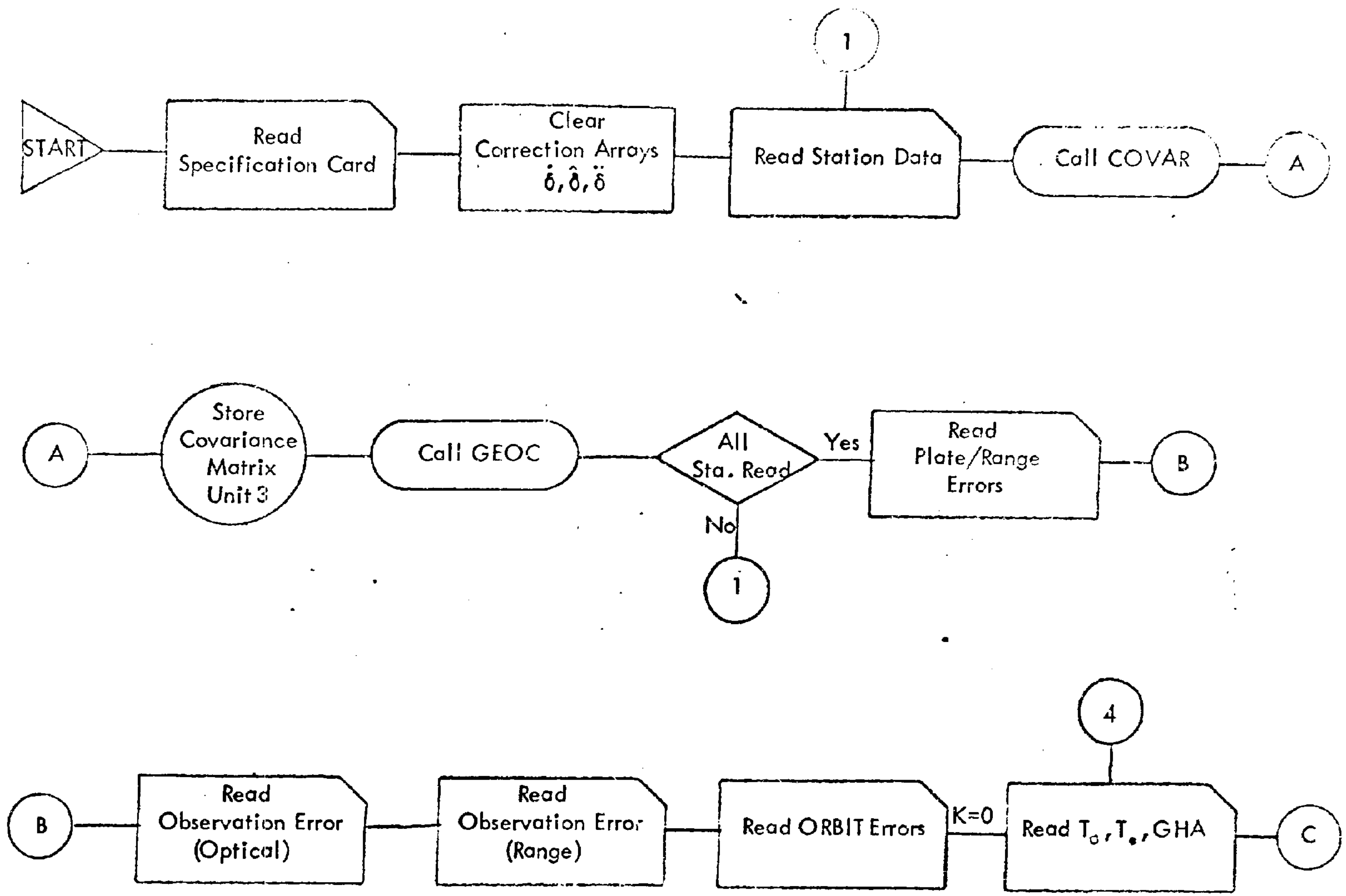
GENERAL DATA FLOW

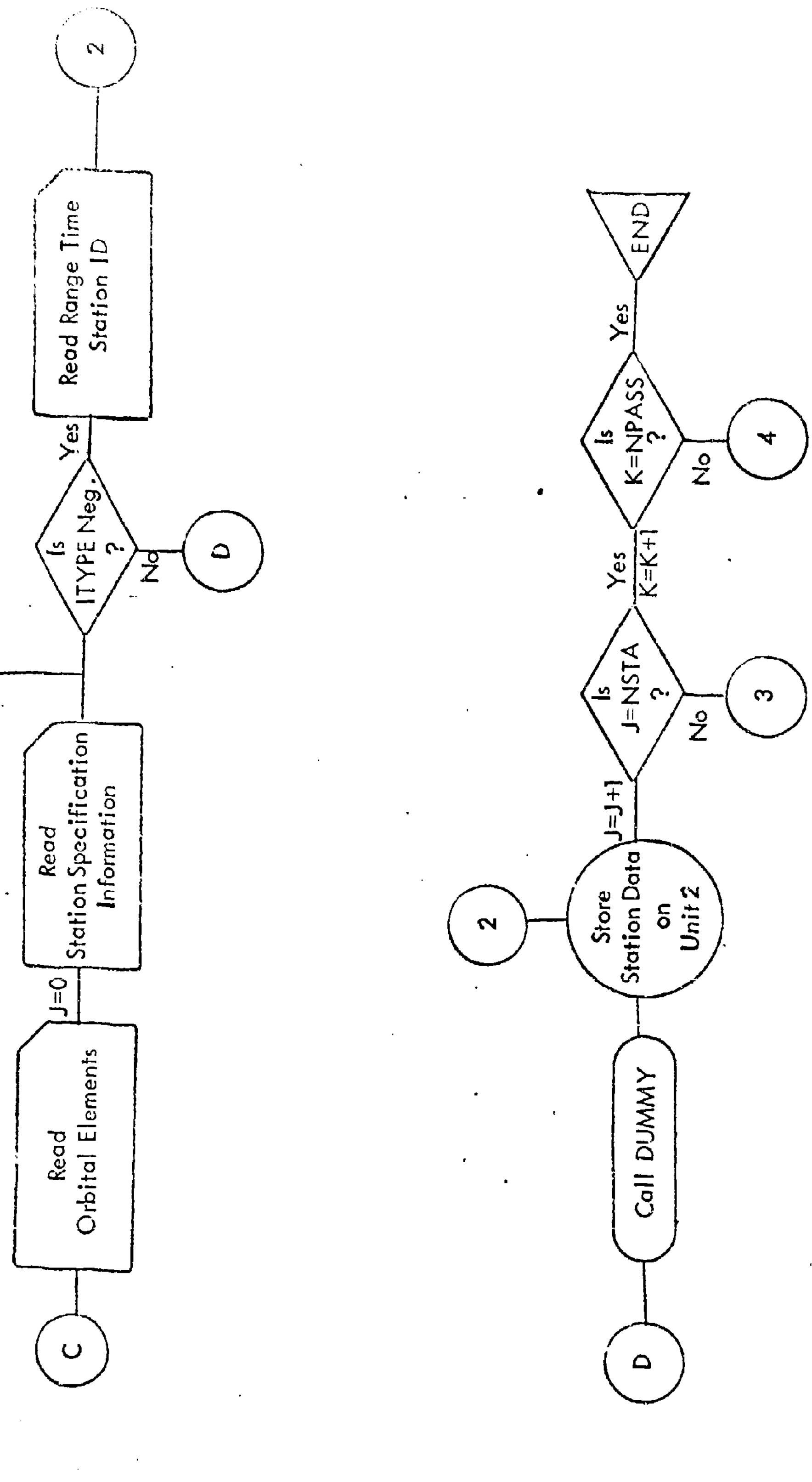


MASTER

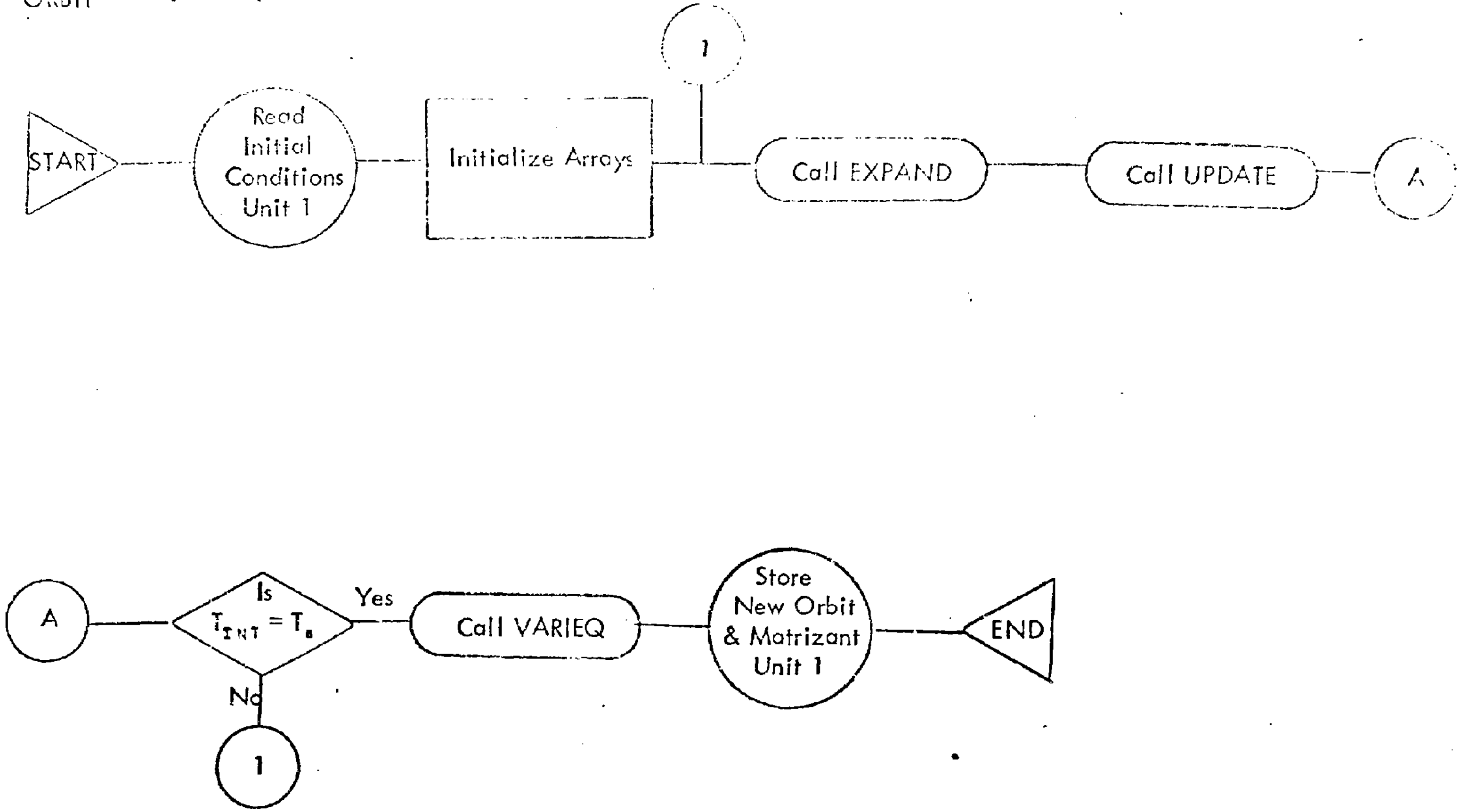

START
Read Baseline ID, Associated Errors For Pair
Read Station Coordinates For Pair
Call GEOC
Form U Matrix (4,6)
A
A
A = UᵀU (6,6)
Form ϵ
C = Uᵀ ϵ (6x1)
Call ATRACT
B
B
Store A and C Unit 3
END

PREP

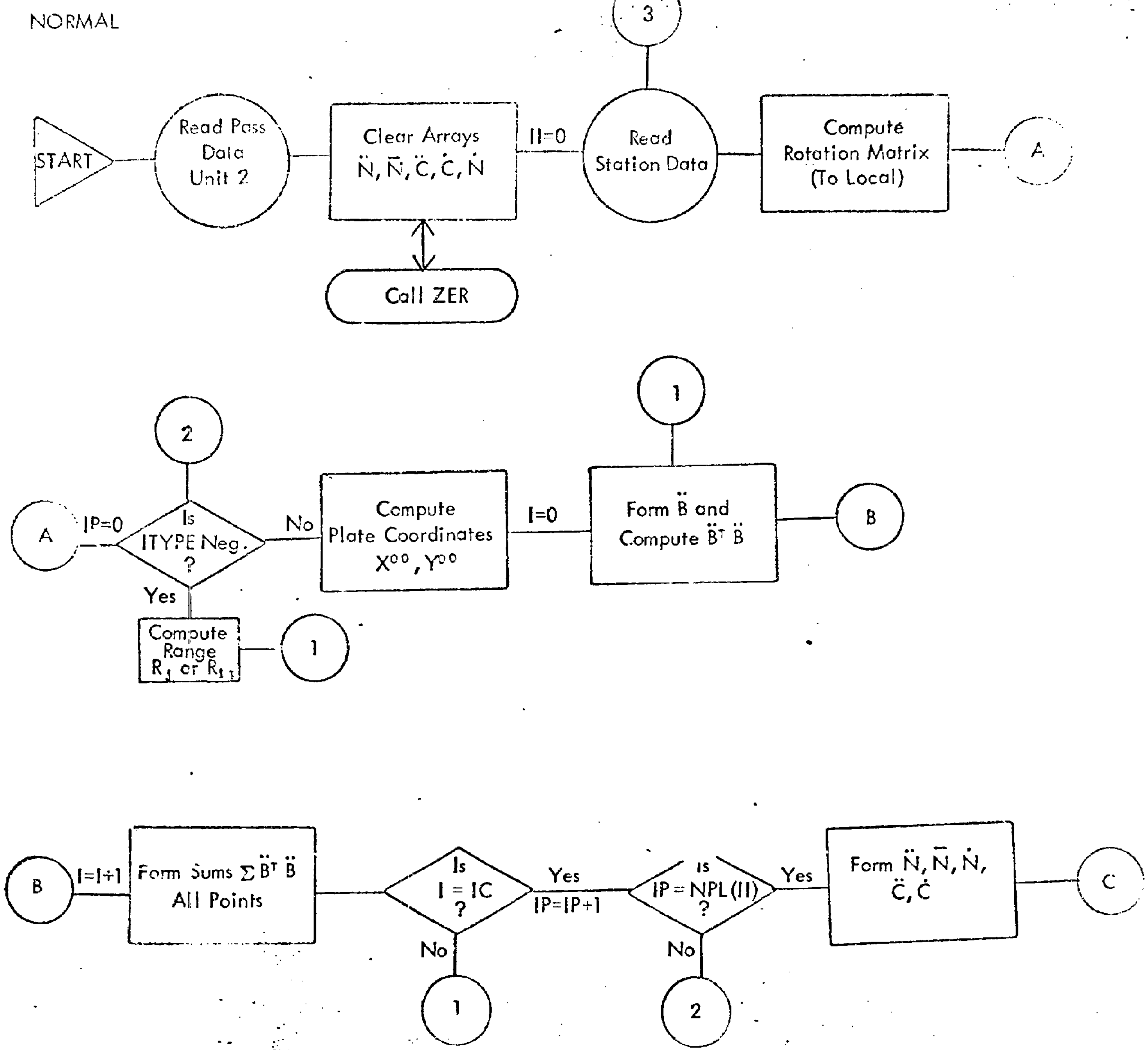

START
Read Specification Card
Clear Correction Arrays $\dot{\delta}, \hat{\delta}, \ddot{\delta}$
1
Read Station Data
Call COVAR
A
A
Store Covariance Matrix Unit 3
Call GEOC
All Sta. Read
Yes
No
1
Read Plate/Range Errors
B
B
Read Observation Error (Optical)
Read Observation Error (Range)
Read ORBIT Errors
K=0
4
Read T_o, T_*, GHA
C

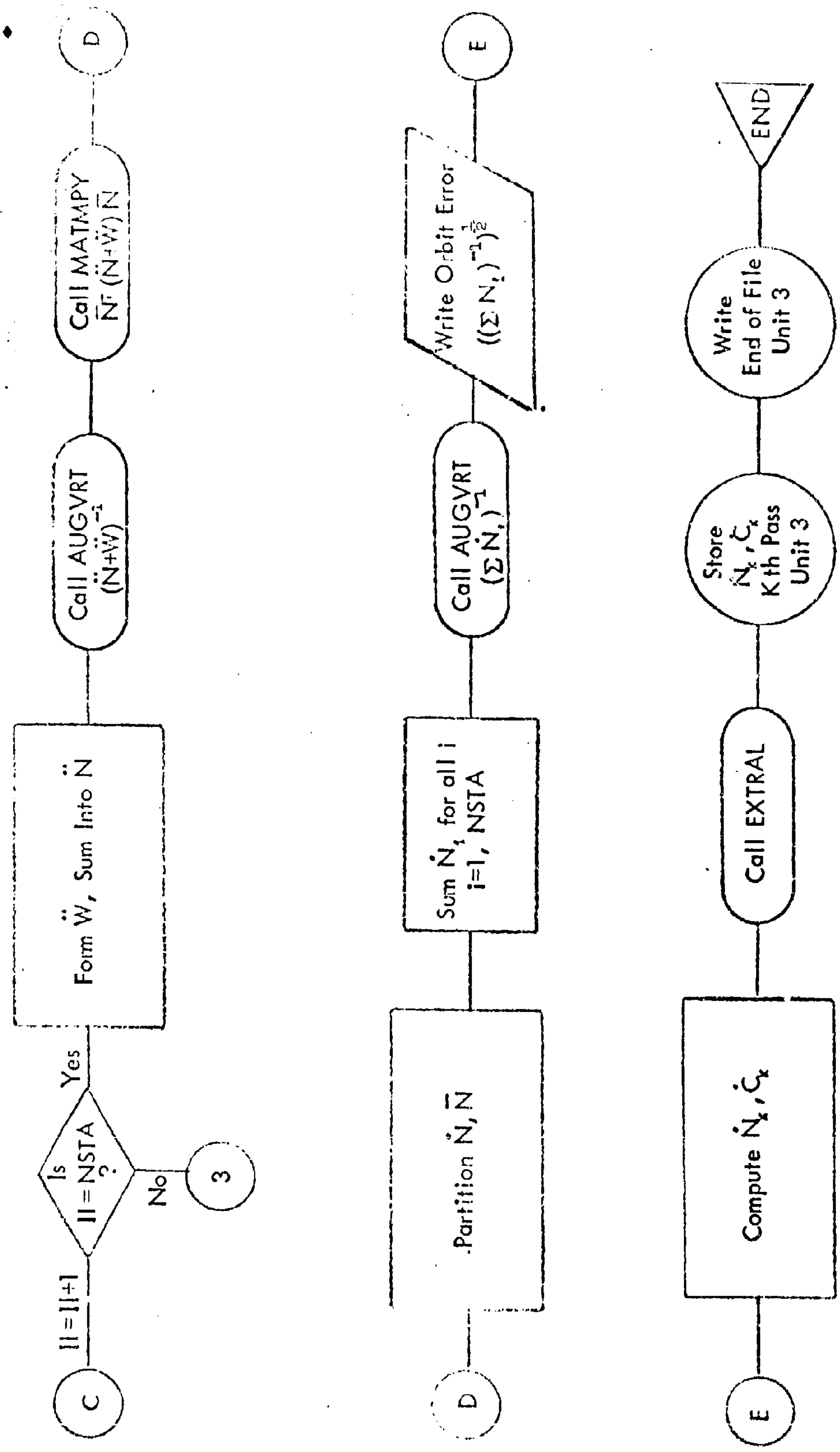


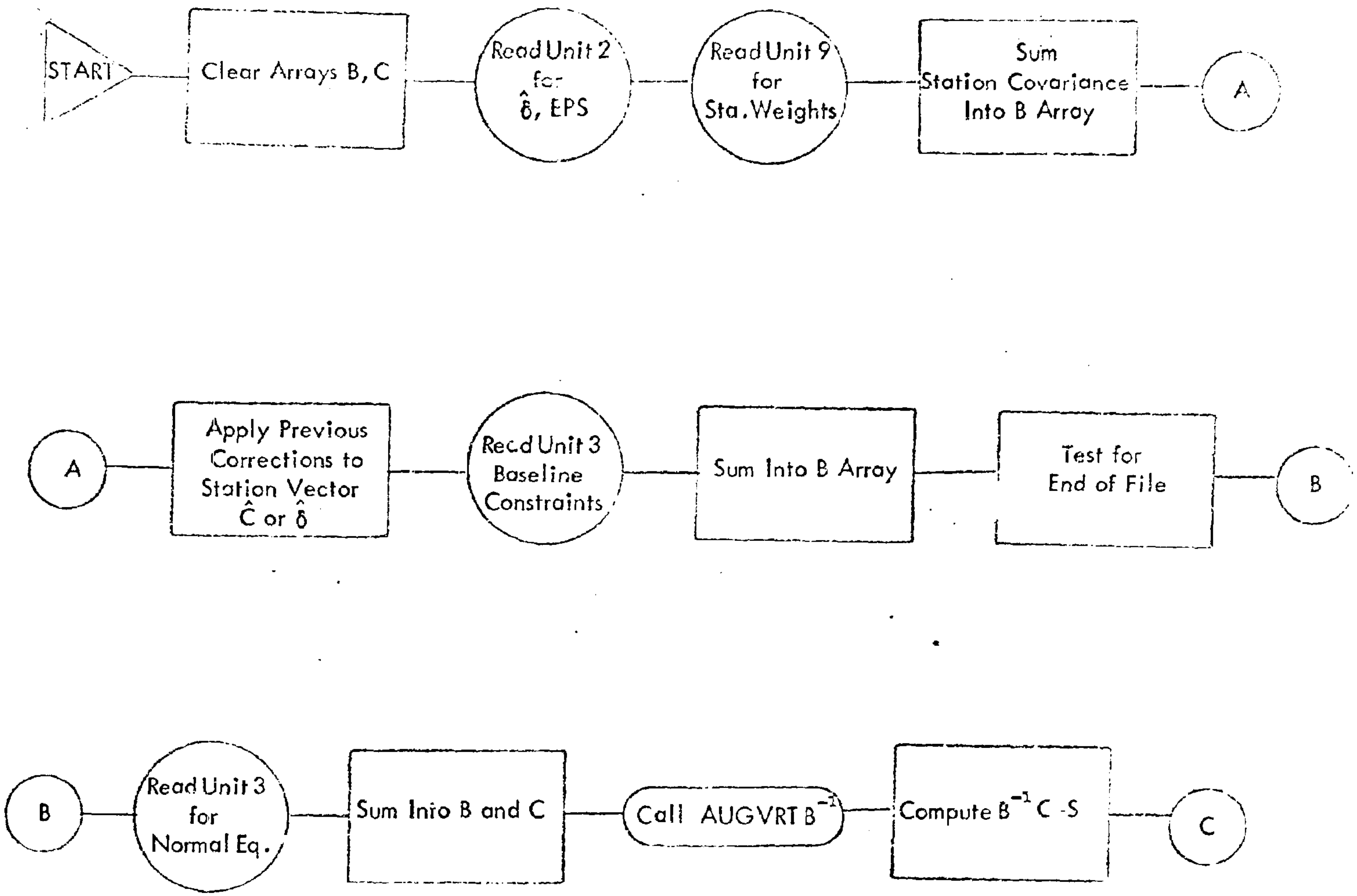
C
Read
Orbital Elements
J=0
Read
Station Specification
Information
3
Is
ITYPE Neg.
?
No
D
Yes
Read Range Time
Station ID
2
D
Call DUMMY
2
Store
Station Data
on
Unit 2
J=J+1
Is
J=NSTA
?
No
3
Yes
K=K+1
Is
K=NPASS
?
No
4
Yes
END

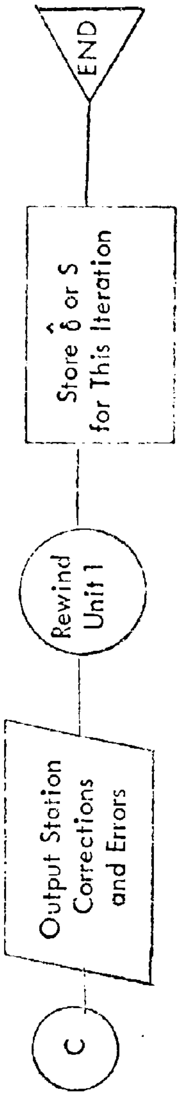

START
Read Initial Conditions Unit 1
Initialize Arrays
1
Call EXPAND
Call UPDATE
A
A
Is $T_{INT} = T_s$
Yes
No
1
Call VARIEQ
Store New Orbit & Matrizant Unit 1
END

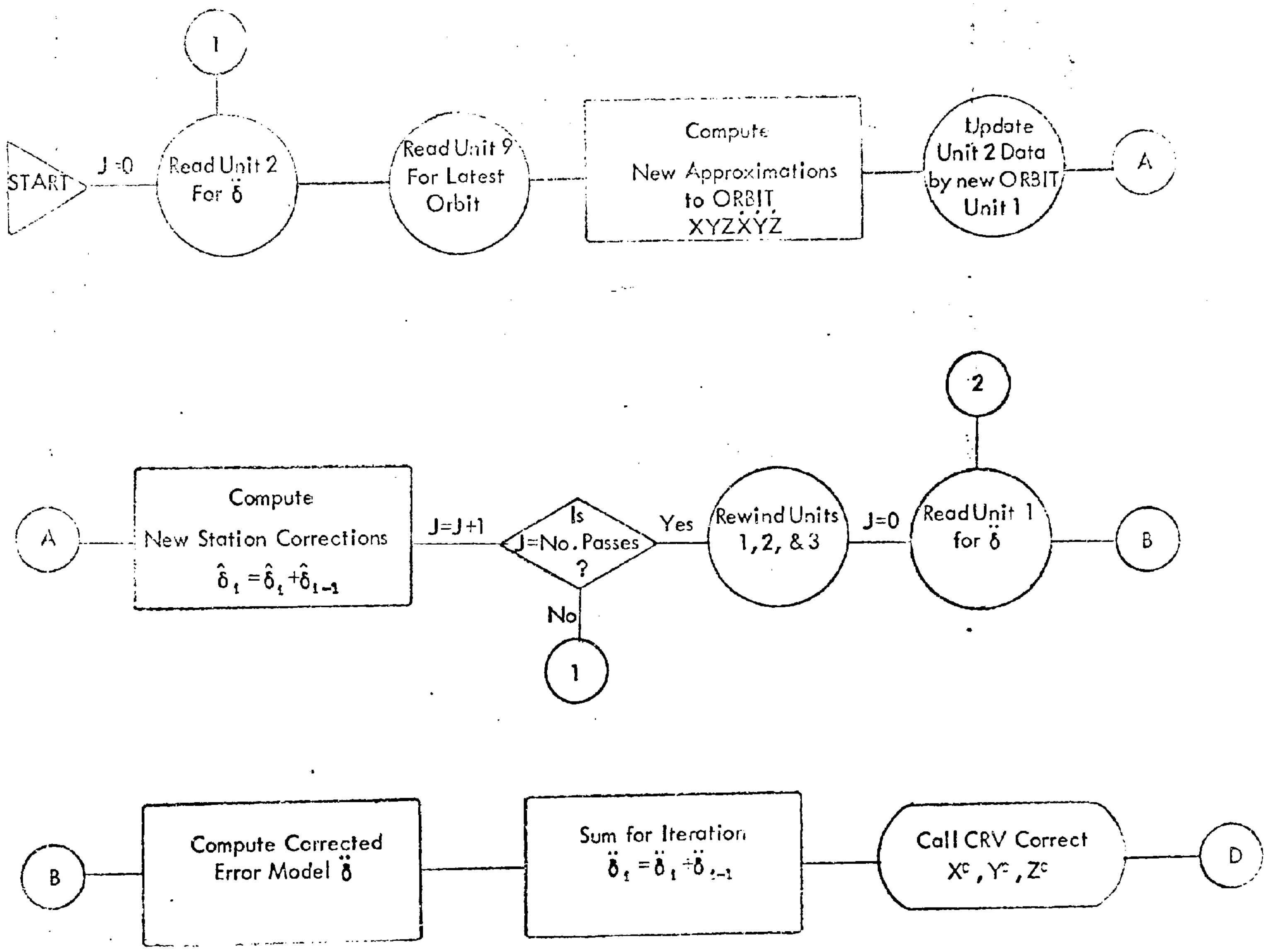
NORMAL

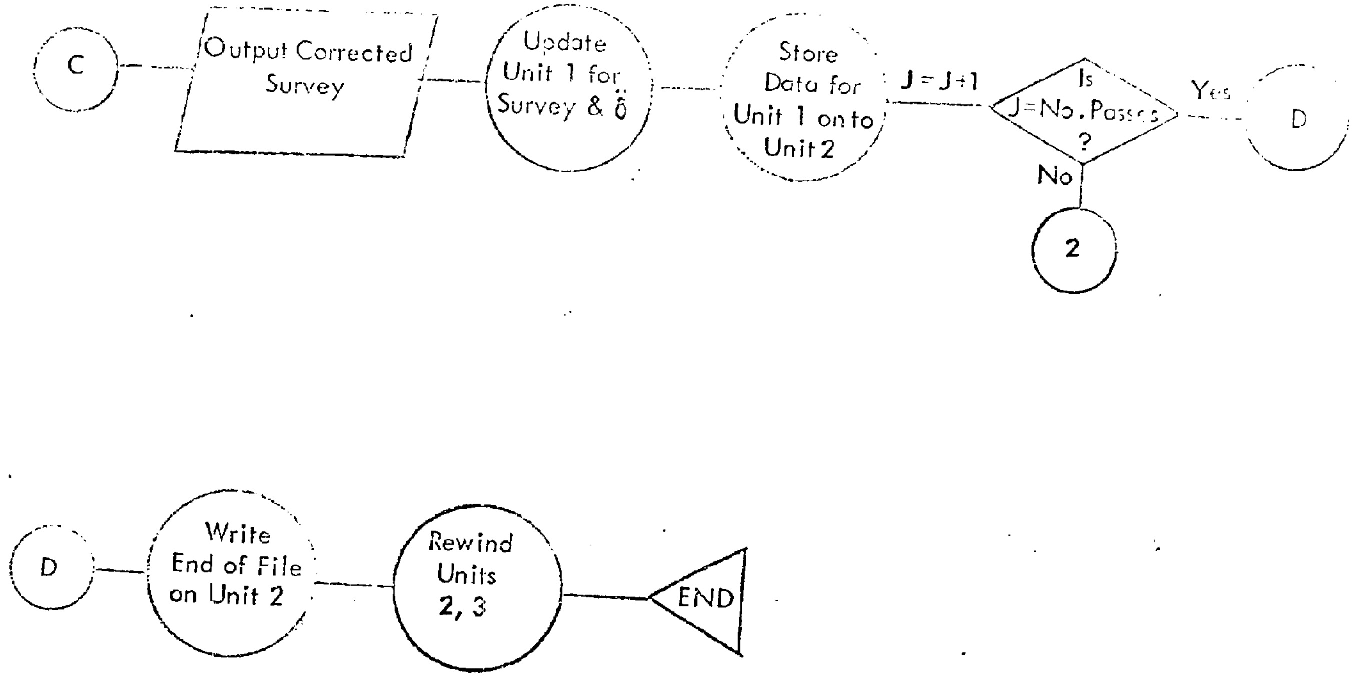

START
Read Pass Data Unit 2
Clear Arrays N̈, N̄, C̈, Ċ, Ṅ
Call ZER
II=0
3
Read Station Data
Compute Rotation Matrix (To Local)
A
A
IP=0
2
Is ITYPE Neg. ?
No
Yes
Compute Range R1 or R11
1
Compute Plate Coordinates X°°, Y°°
I=0
1
Form B̈ and Compute B̈ᵀ B̈
B
B
I=I+1
Form Sums Σ B̈ᵀ B̈ All Points
Is I = IC ?
No
1
Yes
IP=IP+1
Is IP = NPL(II) ?
No
2
Yes
Form N̈, N̄, Ṅ, C̈, Ċ
C


C
II = II+1
Is II = NSTA ?
Yes
No
3
Form Ẅ, Sum Into N̈
Call AUGVRT (N̈+Ẅ)^-1
Call MATMPY N̄ᵀ (N̈+Ẅ) N̄
D
Partition Ṅ, N̄
Sum Ṅᵢ for all i
i=1, NSTA
Call AUGVRT (Σ Ṅᵢ)^-1
Write Orbit Error
((Σ Ṅᵢ)^-1)^½
E
Compute Ṅₖ, Ċₖ
Call EXTRAL
Store Ṅₖ, Ċₖ
K th Pass
Unit 3
Write
End of File
Unit 3
END


START
Clear Arrays B, C
Read Unit 2 for $\hat{\delta}$, EPS
Read Unit 9 for Sta. Weights
Sum Station Covariance Into B Array
A
A
Apply Previous Corrections to Station Vector $\hat{C}$ or $\hat{\delta}$
Read Unit 3 Baseline Constraints
Sum Into B Array
Test for End of File
B
B
Read Unit 3 for Normal Eq.
Sum Into B and C
Call AUGVRT B^{-1}
Compute B^{-1} C - S
C


C
Output Station Corrections and Errors
Rewind Unit 1
Store $\hat{\delta}$ or S for This Iteration
END


START
J=0
Read Unit 2
For δ̈
1
Read Unit 9
For Latest
Orbit
Compute
New Approximations
to ORBIT
XYZẊẎŻ
Update
Unit 2 Data
by new ORBIT
Unit 1
A
A
Compute
New Station Corrections
δ̂ₜ = δ̂ₜ + δ̂ₜ₋₁
J=J+1
Is
J=No. Passes
?
Yes
No
1
Rewind Units
1, 2, & 3
J=0
Read Unit 1
for δ̈
2
B
B
Compute Corrected
Error Model δ̈
Sum for Iteration
δ̈ₜ = δ̈ₜ + δ̈ₜ₋₁
Call CRV Correct
Xᶜ, Yᶜ, Zᶜ
D


C
Output Corrected Survey
Update Unit 1 for Survey & $\ddot{\theta}$
Store Data for Unit 1 onto Unit 2
J = J+1
Is J=No. Passes ?
Yes
No
D
2
D
Write End of File on Unit 2
Rewind Units 2, 3
END

APPENDIX A

Pre Processor

A. General Description

The Pre Processor (PREPRO) program was designed as a supplement to SAGA. It makes corrections to data for polar motion, parallactic refraction, time corrections (UTC-UT1 or SAO-UT1), and phase angle. At option the data is fit to a series of polynomials (2nd thru 6th order) for random measurement residuals evaluation.

In addition to optical corrections, PREPRO also converts GEOCEIVER observations (Doppler counts) to range differences corrected for ionospheric refraction. The residual option also exist for range measurements.

Care must be taken in the application of the admissible preprocessor corrections so that corrections that have already been made are not duplicated. Thus the fact that PC-1000 data processed to date by ACIC have not been corrected for polar motion, parallactic refraction or for UTC to UT1 does not preclude that at some time in the future ACIC policy may change in this regard. When and if it does, the corrections should no longer be applied in the Pre Processor. Thus one should remain up to date on the policies of the various agencies that provide data. In the table below it is indicated which corrections should be applied in the Pre Processor to the various types of data as of January 1969. Though not listed in the table, corrections for phase of optically observed spherical passive satellites can also be generated on option by the Pre Processor. Corrections, when needed, for tropospheric ranging refraction are applied in the main program rather than in the Pre Processor.

TABLE 1

Corrections Remaining to be Applied by Pre-Processor (as of January 1969)

	Polar Motion	Parallactic Refraction	UTC-UT1	A1-UT1	Ionospheric Refraction
PC-1000	X	X	X		
MOTS	X		X		
BAKERNUNN				X	
BC-4		X			
SECOR					
GEOCEIVER					X
LASER					

8. Input

The user selects from Table I the corrections desired. Non-zero values are inserted in the first card corresponding to the particular correction. For each correction desired the follow-up input cards furnish needed input parameters to accomplish the correction. These are outlined according to each correction.

Specification Card

Card No.	Name	Field	Columns	Units	Description
1	ITPE	I4	1- 4		Type of measurement (Range or optic)
	IPOL	I4	5- 8		Polar motion
	IREF	I4	9-12		Parallactic refraction
	ICOR	I4	13-16		UTC to UT1 time
	IRES	I4	17-20		Residual computation
	IPHS	I4	21-24		Phase angle correction
	ISAO	I4	25-28		SAO to UT1 correction

ITPE = 1
Optical Data

Card No.	Name	Field	Columns	Units	Description
2	H	F4.0	1- 4	Degrees	Station latitude
	B	F4.0	5- 8	Minutes	
	D	F7.0	9-15	Seconds	
	T	F7.0	16-22	Degrees	Station longitude
	Z	F7.0	23-29	Minutes	
	V	F7.0	30-36	Seconds	
	HE	F7.0	37-43	Meters	Station height
	F	F9.0	44-52	Meters	Focal length
	IDD	I4	53-56		Station identification

Card No.	Name	Field	Columns	Units	Description
3	H	F4.0	1- 4	Degrees	Greenwich hour angle
	O	F4.0	5- 8	Minutes	For day of interest
	S	F7.0	9-15	Seconds	
	SHT	F7.0	16-22	Nautical Miles	Satellite height
	COR	F7.0	23-29	Seconds	UTC-UT1 correction
	XAN	F7.0	30-36	Arc seconds	x polar angle
	YAN	F7.0	37-43	Arc seconds	y polar angle
	SCAL	F9.0	44-52		Scale on residuals

Insert Only If ISAO $\neq 0$

4A	TAU	F4.0	1- 4	Years	Fraction of tropical year
	TZ	F4.0	5- 8		Epoch of equinox
	DEL	F7.0	9-15	Radians	Nutation in longitude
	OBLQ	F7.0	16-22	Radians	Obliquity of the ecliptic
	GTD	F7.0	23-29	Seconds	A1 time minus UTC time

Insert Only If IPHS $\neq 0$

4B	RADI	F10.0	1-10	Meters	Radius of satellite
	HEGT	F10.0	11-20	Meters	Height of satellite
	RAS	F10.0	21-30	Degrees	Right ascension of sun
	DES	F10.0	31-40	Degrees	Declination of sun

5	DATA - GEOS Format
6	Blank

Card No.	Name	Field	Columns	Units	Description

ITPE = 2
Range Data

Follow specification card with GEOS formatted range data. Follow the data with a blank card to terminate the set.

ITPE = 3
GEOCEIVER Data

Card No.	Name	Field	Columns	Units	Description
2	CS	F10.0	1-10	Mts/sec	Light speed
	FO	F10.0	11-20	Cyc/sec	GEOCEIVER reference frequency
	FS	F10.0	21-30	Cyc/sec	Transmitted frequency
3	ID	I10	1-10		Station identification
	X(I)	F18.9	11-28		Time
	Y(I)	F18.9	29-47		Doppler count measurement
	RR	F18.9	48-65		Refractive Doppler count
4	Blank				

C. Output

The output is in punched card form. The format is compatible with SAGA input. The only written output is the residuals of the 2nd thru 6th order polynomials and conic.

Written output of the corrected observations may be obtained by changing the file unit number to the appropriate output file number.

D. Analysis

Polar Motion Correction

Let:

T = time of observation in years and days (e.g., T=1966.385)

t = time of observation from ZULU midnight

α, δ = right ascension observed

x, y = angles of polar motion (radians).

Compute:

(1) θ = apparent Greenwich sidereal time at time t of observation

$$= \theta_0 + \omega t, \text{ where } \theta_0 = \text{Greenwich hour angle}$$
$$\omega = \text{earth's rotational rate}$$

(2) In terms of given α, δ and computed value of θ, x, y evaluate the expression

$$\begin{Bmatrix} \lambda \\ \mu \\ \nu \end{Bmatrix} = \begin{Bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & 0 & x \\ 0 & 1 & -y \\ -x & y & 1 \end{Bmatrix} \begin{Bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{Bmatrix}$$

(3) Compute α', δ' from the relations

$$\sin\alpha' = \mu / \sqrt{1-\nu^2}$$
$$\cos\alpha' = \lambda / \sqrt{1-\nu^2}$$
$$\sin\delta' = \nu \text{ (sign of } \delta' \text{ is same as sign of } \nu)$$

(4) Replace α, δ by α', δ'.

Phase Angle Correction

The apparent direction of a spherical, sun reflecting, satellite may be biased, depending upon the portion of surface observed. Two different types of light reflections are encountered. First is that of diffused reflection, which is the result of reflections from irregular surfaces. The moon is an example of diffused reflection. Second is that of specular reflections, the result of light reflected from smooth surfaces such as a mirror.

In the case of non-spherical satellites, the correction is treated as a constant bias and γ (section 4). An attempt to apply direction corrections would be nearly impossible and certainly impractical because of various satellite configurations and attitudes.

An approximate time of observation is required in determining the direction of the sun from the earth's center. The time of either the pre- or post-calibration is sufficient. If both are available the mean of the two may be taken as the time of observation.

A. Diffused Reflection

Compute the direction cosines (inertial) of the sun for time of observation, given right ascension (α), declination (δ) and time.

$$(1)\qquad \begin{aligned} \lambda &= \cos\alpha\cos\delta \\ \mu &= \sin\alpha\cos\delta \\ \nu &= \sin\delta. \end{aligned}$$

Convert to earth-fixed direction cosines

$$(2)\qquad \begin{bmatrix} \lambda_s \\ \mu_s \\ \nu_s \end{bmatrix} = \begin{bmatrix} \cos\theta_g & \sin\theta_g & 0 \\ \cos\theta_g & -\sin\theta_g & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix}$$

where

$$(3)\qquad \theta_g = \theta_{g_0} + \dot{\psi}(t_0)$$

θ_{g_0} = right ascension of Greenwich for year of interest

$\dot{\psi}$ = rotation rate of earth in radians/day

t_0 = time in days and decimal days from January 0.

Let

$$(4)\qquad V_1 = \lambda_s \underline{i} + \mu_s \underline{j} + \nu_s \underline{k}$$

be the direction vector from the earth's center to the sun. Assume V_1 is parallel to the vector from the satellite to the sun.

Compute $\lambda_{ij}, \mu_{ij}, \nu_{ij}$ from α, δ to satellite and let

(5) $\quad V_{2ij} = \lambda_{ij}\underline{i} + \mu_{ij}\underline{j} + \nu_{ij}\underline{k}$

be the jth direction vector from station i to the satellite.

The dot product of $(+V_1)$ and $(-V_2)$ yields the intersection angle at the satellite

(6) $\quad \eta_{ij} = \cos^{-1}\left((+V_1)(-V_{2ij})\right)$.

The distance d_{ij} normal to V_{2ij} from the satellite center is inversely proportional to the percentage of the illuminated surface viewed by camera i. Solve for d_{ij}:

(7) $$\frac{d_{ij}}{r_s} = \left(1 - \frac{\pi - \eta_{ij}}{\pi}\right), \qquad d'_{ij} = \frac{d_{ij}}{\sin \eta_{ij}}$$

where r_s is the satellite radius and d'_{ij} has the same direction as $-V_1$. The correction vector $\overline{V}_{1ij}$ is

(8) $\quad \overline{V}_{1ij} = d'_{ij}\lambda_s\underline{i} + d'_{ij}\mu_s\underline{j} + d'_{ij}\nu_s\underline{k}$

The magnitude of $\overline{V}_{2ij}$ is:

(9) $$R_{ij} = \left[(X_j^c - X_i^c)^2 + (Y_j^c - Y_i^c)^2 + (Z_j^c - Z_i^c)^2\right]^{\frac{1}{2}}.$$

Then $\overline{V}_2$ is:

(10) $\quad \overline{V}_{2ij} = R_{ij}\lambda_{ij}\underline{i} + R_{ij}\mu_{ij}\underline{j} + R_{ij}\nu_{ij}\underline{k}$.

The vector $\overline{V}_3$ from station i to the jth satellite center is:

(11) $\quad \overline{V}_{3ij} = \overline{V}_{1ij} + \overline{V}_{2ij}$.

Compute the $\underline{i}$, $\underline{j}$, and $\underline{k}$ components of $\overline{V}_{B ij}$:

$$(12) \begin{pmatrix} S_{1ij} \\ S_{2ij} \\ S_{3ij} \end{pmatrix} = \begin{pmatrix} \lambda_s & \lambda_{ij} \\ \mu_s & \mu_{ij} \\ \nu_s & \nu_{ij} \end{pmatrix} \begin{pmatrix} d'_{ij} \\ R_{ij} \end{pmatrix}$$

$$(13) \quad \overline{V} = (S_{1ij}^2 + S_{2ij}^2 + S_{3ij}^2)^{\frac{1}{2}} .$$

The new direction cosines corrected to the center of the satellite are:

$$(14) \begin{pmatrix} \lambda_{ij} \\ \mu_{ij} \\ \nu_{ij} \end{pmatrix} = \frac{1}{\overline{V}} \begin{pmatrix} S_{1ij} \\ S_{2ij} \\ S_{3ij} \end{pmatrix} .$$

Replace λ_{ij}, μ_{ij}, ν_{ij} in Appendix B with (14) and convert to new $\alpha\,\delta$. the triangulation except the residual computations.

B. <u>Specular Reflection</u>

The only difference from the correction in (A) is the computation of d'_{ij}. Since we assume a truly spherical satellite, the basic laws of reflectivity hold (the angle of reflection equals the angle of incidence). The solution of d'_{ij} becomes a simple relation of the law of sines and cosines.

Compute:

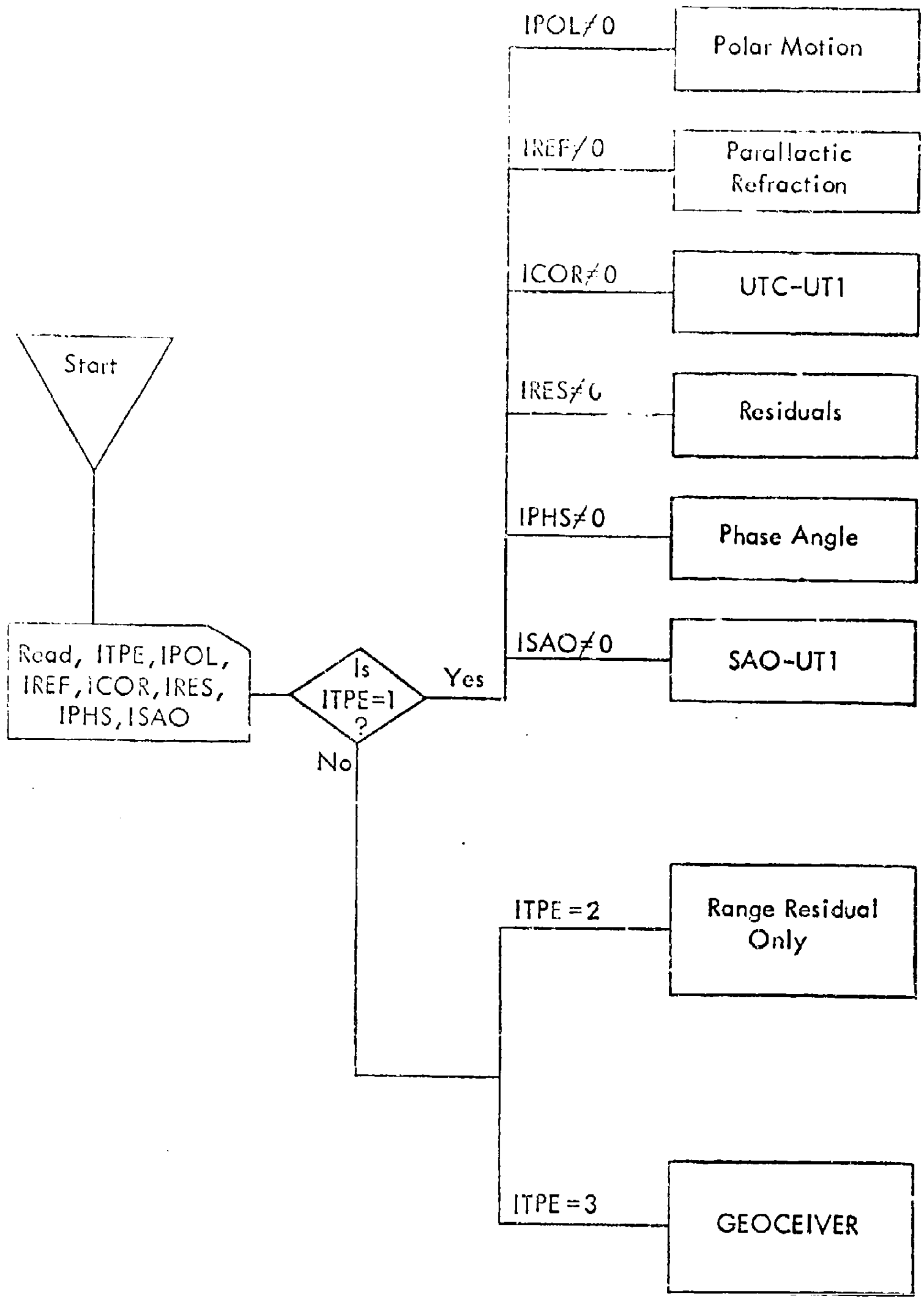
$$(1) \quad d'_{ij} = r_s \sin \alpha_{ij} / \sin(\pi - 2\,\alpha_{ij})$$

where

$$\alpha_{ij} = \eta_{ij}/2.$$

GEOCEIVER Range Difference Computation

See Section 3.0 of Part I for a detailed analysis of the GEOCEIVER range difference reductions and the formation of the basic error model.



Pre-Processor Program

APPENDIX B

Dummy Camera Projection

Optical observations are converted from right ascension and declination to plate coordinates (x, y). First convert right ascension and declination to azimuth (A_j) and elevation (E_j). Assume the middle observation (azimuth and elevation) to be the direction of the principal axis of the camera. The external orientation elements (α, ω, κ) become:

$$\alpha = A_m, \quad \omega = E_m \text{ and let } \kappa = 0.$$

For the i th station and time t_j compute the local hour angle of the j th observation.

$$t_s = t_o + 1.00273791\, t_j - \lambda - \alpha$$

where

t_o = Greenwich hour angle at $t = 0$ (apparent)

t_j = observing time (universal) in angular measurement from Greenwich (radians)

λ = longitude of station (West)

α = right ascension of observation

Φ = latitude of station

then

$$\sin E = \sin \Phi \sin \delta + \cos \Phi \cos \delta \cos (t_s)$$

$$\cos E = (1 - \sin E)^{\frac{1}{2}}$$

$$\sin A = -\cos \delta \sin (t_s)/\cos E$$

$$\cos A = (\sin \delta - \sin \Phi \sin E)/(\cos \Phi \cos E)$$

then

$$A_j = \tan^{-1} (\sin A/\cos A)$$

$$E_j = \tan^{-1} (\sin E/\cos E)$$

use midpoint A_j, E_j to compute the local orientation matrix.

$$\begin{pmatrix} \hat{A} & \hat{B} & \hat{C} \\ \hat{A}' & \hat{B}' & \hat{C}' \\ \hat{D} & \hat{E} & \hat{F} \end{pmatrix}_j = \begin{pmatrix} -\cos\alpha & \sin\alpha & 0 \\ -\sin\alpha\sin\omega & -\cos\alpha\sin\omega & \cos\omega \\ \sin\alpha\cos\omega & \cos\alpha\cos\omega & \sin\omega \end{pmatrix}_j$$

then

$$\begin{pmatrix} \ell \\ m \\ n \end{pmatrix}_j = \begin{pmatrix} \sin A \;\; \cos E \\ \cos A \;\; \cos E \\ \sin E \end{pmatrix}_j$$

and

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_j = \begin{pmatrix} \hat{A} & \hat{B} & \hat{C} \\ \hat{A}' & \hat{B}' & \hat{C}' \\ \hat{D} & \hat{E} & \hat{F} \end{pmatrix} \begin{pmatrix} \ell \\ m \\ n \end{pmatrix}_j$$

the plate coordinates for the j th point are:

$$\hat{x}_j = cu_j/w_j$$

$$\hat{y}_j = cv_j/w_j.$$

Let $(\hat{x}_a, \hat{y}_a)$, $(\hat{x}_b, \hat{y}_b)$ denote the coordinates of the first and last points on the trace. Let $\varkappa$ denote the angle between the line joining these two points and the $\hat{x}$ axis. Then compute:

$$\sin\varkappa = -(\hat{y}_b - \hat{y}_a)/r_{ab}$$

$$\cos\varkappa = (\hat{x}_b - \hat{x}_a)/r_{ab}$$

$$r_{ab} = [(\hat{x}_b - \hat{x}_a)^2 + (\hat{y}_b - \hat{y}_a)^2]^{\frac{1}{2}}$$

The new x, y coordinates are:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\cos \varkappa & \sin \varkappa \\ \sin \varkappa & \cos \varkappa \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}_j .$$

The rotation of the local orientation into the master frame is accomplished by:

$$\begin{bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{bmatrix} = \begin{bmatrix} -\cos \varkappa & \sin \varkappa & 0 \\ \sin \varkappa & \cos \varkappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} & \hat{C} \\ \hat{A}' & \hat{B}' & \hat{C}' \\ \hat{D} & \hat{E} & \hat{F} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -\sin \varphi & 0 & \cos \varphi \\ \cos \varphi & 0 & \sin \varphi \end{bmatrix} \begin{bmatrix} \cos \lambda & -\sin \lambda & 0 \\ \sin \lambda & \cos \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

The coordinates x_j, y_j and the master orientation matrix serve as input to the main program.

APPENDIX C

Interstation Constraints

A. Incorporation of Special Interstation Constraints

From the latest approximations to the coordinates of the j th specified station pair p_j, q_j compute (subscripts j are omitted in following):

$$\Delta X_{pq} = (X_p^c)^{oo} - (X_q^c)^{oo}$$

(1) $$\Delta Y_{pq} = (Y_p^c)^{oo} - (Y_q^c)^{oo}$$

$$\Delta Z_{pq} = (Z_p^c)^{oo} - (Z_q^c)^{oo}$$

$$R_{pq}^{oo} = [(\Delta X_{pq})^2 + (\Delta Y_{pq})^2 + (\Delta Z_{pq})^2]^{\frac{1}{2}}$$

$$r_{pq} = [(\Delta X_{pq})^2 + (\Delta Y_{pq})^2]^{\frac{1}{2}}$$

$$\sin A_{pq}^{oo} = \Delta Y_{pq}/r_{pq}^{oo}, \quad A_{pq}^{oo} = \text{arc sin } (\sin A_{pq}^{oo})$$

$$\sin E_{pq}^{oo} = \Delta Z_{pq}/R_{pq}^{oo}, \quad E_{pq}^{oo} = \text{arc sin } (\sin E_{pq}^{oo})$$

$$\lambda_{pq} = \Delta X_{pq}/R_{pq}^{oo}$$

$$\mu_{pq} = \Delta Y_{pq}/R_{pq}^{oo}$$

$$\nu_{pq} = \Delta Z_{pq}/R_{pq}^{oo}.$$

Compute:

(2)

$$a_{11} = \frac{1}{\sigma_{A_{pq}}} \frac{\partial A_{pq}^{00}}{\partial X_p} = -\frac{1}{\sigma_{A_{pq}}} \frac{\cos A_{pq}^{00}}{r_{pq}} \qquad b_{11} = \frac{1}{\sigma_{E_{pq}}} \frac{\partial E_{pq}^{00}}{\partial X_p} = -\frac{1}{\sigma_{E_{pq}}} \frac{(\lambda_{pq} \sin E_{pq}^{00})}{r_{pq}}$$

$$a_{12} = \frac{1}{\sigma_{A_{pq}}} \frac{\partial A_{pq}^{00}}{\partial Y_p} = -\frac{1}{\sigma_{A_{pq}}} \frac{\sin A_{pq}^{00}}{r_{pq}} \qquad b_{12} = \frac{1}{\sigma_{E_{pq}}} \frac{\partial E_{pq}^{00}}{\partial Y_p} = -\frac{1}{\sigma_{E_{pq}}} \frac{(\mu_{pq} \sin E_{pq}^{00})}{r_{pq}}$$

$$a_{13} = \frac{1}{\sigma_{A_{pq}}} \frac{\partial A_{pq}^{00}}{\partial Z_p} = 0 \qquad b_{13} = \frac{1}{\sigma_{E_{pq}}} \frac{\partial E_{pq}^{00}}{\partial Z_p} = -\frac{1}{\sigma_{E_{pq}}} \frac{(1 - \nu_{pq} \sin E_{pq}^{00})}{r_{pq}}$$

$$c_{11} = \frac{1}{\sigma_{R_{pq}}} \frac{\partial R_{pq}^{00}}{\partial X_p} = \frac{1}{\sigma_{R_{pq}}} \lambda_{pq}$$

$$c_{12} = \frac{1}{\sigma_{R_{pq}}} \frac{\partial R_{pq}^{00}}{\partial Y_p} = \frac{1}{\sigma_{R_{pq}}} \mu_{pq}$$

$$c_{13} = \frac{1}{\sigma_{R_{pq}}} \frac{\partial R^{00}}{\partial Z_p} = \frac{1}{\sigma_{R_{pq}}} \nu_{pq} .$$

When a prespecified linear constraint is to be imposed on the coordinates of stations p and q compute:

(3) $U_{pq}^{00} = \alpha_1 (X_p^c)^{00} + \alpha_2 (Y_p^c)^{00} + \alpha_3 (Z_p^c)^{00} + \beta_1 (X_p^c)^{00} + \beta_2 (Y_p^c)^{00} + \beta_3 (Z_p^c)^{00}$

$$d_{11} = \frac{1}{\sigma} \frac{\partial U^{00}}{\partial X_p^c} = \frac{1}{\sigma} \alpha_1 \qquad d_{14} = \frac{1}{\sigma} \frac{\partial U^{00}}{\partial X_q^c} = \frac{1}{\sigma} \beta_1$$

$$d_{12} = \frac{1}{\sigma} \frac{\partial U^{00}}{\partial Y_p^c} = \frac{1}{\sigma} \alpha_2 \qquad d_{15} = \frac{1}{\sigma} \frac{\partial U^{00}}{\partial Y_q^c} = \frac{1}{\sigma} \beta_2$$

$$d_{13} = \frac{1}{\sigma} \frac{\partial U^{00}}{\partial Z_p^c} = \frac{1}{\sigma} \alpha_3 \qquad d_{16} = \frac{1}{\sigma} \frac{\partial U^{00}}{\partial Z_q^c} = \frac{1}{\sigma} \beta_3 .$$

In terms of the above set up:

$$(4)\quad \underset{(4,6)}{U_{pq}} = \begin{array}{c} \left[\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & -a_{11} & -a_{12} & -a_{13} \\ b_{11} & b_{12} & b_{13} & -b_{11} & -b_{12} & -b_{13} \\ c_{11} & c_{12} & c_{13} & -c_{11} & -c_{12} & -c_{13} \\ d_{11} & d_{12} & d_{13} & -d_{11} & -d_{12} & -d_{13} \end{array}\right] \end{array} \begin{array}{l} \underline{\text{Azimuth Constraint}} \\ \underline{\text{Elevation Constraint}} \\ \underline{\text{Distance Constraint}} \\ \underline{\text{Linear Constraint}} \end{array}$$

Let $A^0_{pq}, E^0_{pq}, R^0_{pq}, U^0_{pq}$ denote the measured values and let $\sigma_{A_{pq}}, \sigma_{E_{pq}}\ \sigma_{R_{pq}}\ \sigma_{U_{pq}}$ be the corresponding standard deviations. Then set up the discrepancy vector:

$$\epsilon_{pq} = \begin{bmatrix} (A^{00}_{pq} - A^0_{pq})/\sigma_{A_{pq}} \\ (E^{00}_{pq} - E^0_{pq})/\sigma_{E_{pq}} \\ (R^{00}_{pq} - R^0_{pq})/\sigma_{R_{pq}} \\ (U^{00}_{pq} - U^0_{pq})/\sigma_{U_{pq}} \end{bmatrix}.$$

If a particular type of constraint is not to be exercised between points p and q, the rows of U_{pq} and ϵ_{pq} corresponding to that constraint should be set equal to zero. Thus if a distance constraint were to be rows of U_{pq} and ϵ_{pq} would consist of zero elements. In terms of U_{pq} and ϵ_{pq} compute:

$$\underset{(6\times 6)}{S_{pq}} = U^T_{pq}\, U_{pq}, \quad c_{pq} = U^T_{pq}\, \epsilon_{pq}\ .$$

Partition S_{pq} and c_{pq} as follows:

$$(5)\quad S_{pq} = \begin{bmatrix} \underset{(3,3)}{\dot{S}_{pp}} & \underset{(3,3)}{\dot{S}_{pq}} \\ \underset{(3,3)}{\dot{S}^T_{pq}} & \underset{(3,3)}{\dot{S}_{qq}} \end{bmatrix}, \quad c_{pq} = \begin{bmatrix} \underset{(3,1)}{\dot{c}_p} \\ \underset{(3,1)}{\dot{c}_q} \end{bmatrix}.$$

(6) Merge S_{pq} and c_{pq} into the proper position of the network normal equation, S and c, respectively.

(7) Continue as above until all interstation constraints have thus been processed.

APPENDIX D

Preliminary Geoceiver Computations

The raw Geoceiver cycle counts ΔN_j are first converted to range differences between the satellite over the time interval t_{j-1} to t_j. If

f_0 = transmitted frequency
f_0' = Geoceiver reference frequency
t_j = time of jth cycle count
s_j = slant range between Geoceiver and satellite
c = velocity of light

then

$$\Delta s_j = s_j - s_{j-1} = \lambda_0 D_j - \lambda_0 (f_0' - f_0)(t_j - t_{j-1})$$

where

$$\lambda_0 = c/f_0$$

and D_j is the cycle count corrected for ionospheric refraction. D_j is given by:

$$D_j = \Delta N_j - K \Delta\Delta N_j$$

where

$\Delta\Delta N_j$ = refractive cycle count
K = 9 for 162-324mc reception
K = 9 1/6 for 150-400mc reception.

The range differences are converted to relative ranges by:

$$r_j = \Delta s_1 + \Delta s_2 + \ldots + \Delta s_j$$

where

r_j = change in range from time $t = t_0$ to time $t = t_j$.

The quantities t_j, r_j constitute the input to the main program. When any particular ΔN_j is equal to zero, this means that phase lock was lost during the jth counting interval and it becomes necessary in the main program to reinitialize zero set at time t_j.

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Bedford, Massachusetts 01730

13. ABSTRACT

The development of the Short Arc Geodetic Adjustment (SAGA) program is outlined in detail. The application of the computer program with actual data illustrates the capabilities of the SAGA geodetic positioning. Complete instructions to the program user are included in the report.

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SUBJECT: Contract F19628-68-C-0093

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The ERRATA was discussed with Alice Healy and denotes the change in the title.

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TECH, OTHER	Air Force Cambridge Research Laboratories (CRJ) L.G. Hanscom Field Bedford, Massachusetts 01730

13. ABSTRACT The program SAGA exploits short arc orbital constraints in effecting the adjustment of observations made by geodetic tracking nets embracing both optical systems (e.g., PC-1000, MOTS) and electronic ranging systems (Lasers, Secor, Geoceiver, etc). Provisions are made for consideration of:

a) random errors in the observations and in the timing of observations;
b) serially correlated errors in observations;
c) errors in the adopted location of the center of mass;
d) systematic errors governed by error models having coefficients subject to a prior constraints.

The overall tracking net can include an indefinitely large number of stations (many hundreds) as long as no more than fifteen participate successfully in the observations of any pass. All orbital state vectors are treated as unknown and no limits are set on the number of state vectors that can be solved for simultaneously. Allowances are made in optical error modelling for reinitialization of error coefficients that becomes necessary when any station exposes more than one plate on a given pass. In the case of electronic tracking, up to three dropouts in tracking can be accommodated for each station on each pass with appropriate reinitialization of error coefficients. A maximum of over 250 error coefficients can be exercised in the reduction of each pass. This becomes computationally feasible by virtue of algorithms providing the solution to problems in what is termed second order partitioned regression.

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